Differential Geometry and Hydrodynamics: more than two Centuries of Interaction

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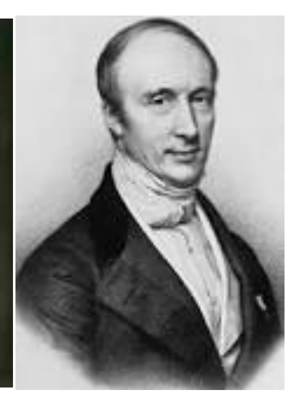
$$\overset{Lagrange}{1781} \longrightarrow \overset{Helmholtz}{1858} \longrightarrow \overset{Clebsch}{1859} \longrightarrow \overset{Hankel}{1861} \longrightarrow \overset{Kelvin}{1869} \longrightarrow \overset{Synge}{1937} \longrightarrow \overset{Arnold}{1966}$$

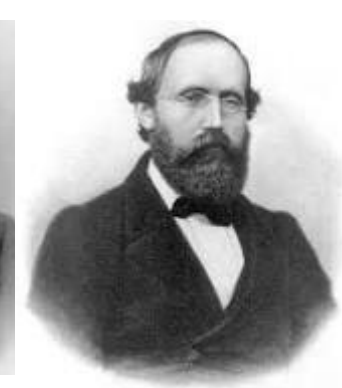

L. Euler
1707-1783


J. le Rond
D'Alembert
1717-1783


J.F. Pfaff
1765-1825


A. Cauchy
1789-1857


CGJ Jacobi
1804-1851


B. Riemann
1826-1866


H. Poincaré
1854-1912


E. Cartan
1869-1951


H. Whitney
1907-1989

Lagrange and steady-state 2D Euler flow (1781)

Not just 2D

Joseph-Louis Lagrange 1736-1813

$$\partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\nabla p, \qquad \nabla\cdot\boldsymbol{v} = 0$$

$$\partial_t \boldsymbol{v} - \boldsymbol{v}\times\boldsymbol{\omega} = -\nabla p_\star, \qquad \nabla\cdot\boldsymbol{v} = 0$$

$$\boldsymbol{\omega} := \nabla\times\boldsymbol{v}, \qquad p_\star := p + (1/2)|\boldsymbol{v}|^2$$

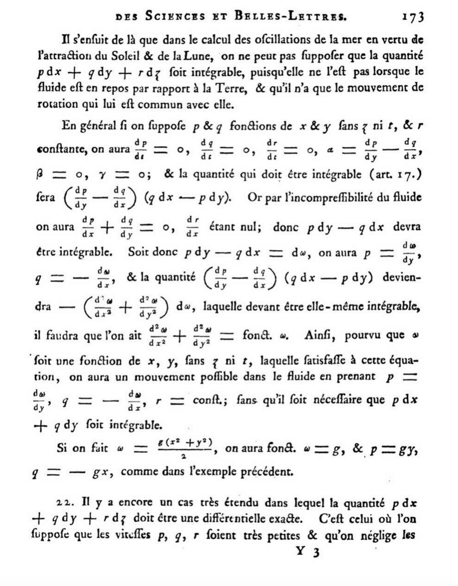
$$\partial_t \boldsymbol{v}\cdot d\boldsymbol{x} - (\boldsymbol{v}\times\boldsymbol{\omega})\cdot d\boldsymbol{x} = -dp_\star, \qquad \nabla\cdot\boldsymbol{v} = 0$$

Steady-state 2D case: $\boldsymbol{v} = (u, v, 0), \quad \boldsymbol{\omega} = (0, 0, \zeta)$

$$\zeta v_\perp = -dp_\star, \quad v_\perp = d\psi \quad v_\perp := (\mathbf{e}_3\times\boldsymbol{v})\cdot d\boldsymbol{x} \ \ \text{(1-form)}$$

Thus $d\zeta\wedge d\psi = 0$. Hence $\zeta = F(\psi)$.

Seven years later in the introduction of his "Analytic Mechanics", Lagrange writes: *On ne trouvera point de Figures dans cet Ouvrage. Les méthodes que j'y expose ne demandent ni constructions, ni raisonnements géométriques ou méchaniques, mais seulement des opérations algébriques, assujetties à une marche régulière & uniforme. Ceux qui aiment l'Analyse, verront avec plaisir la Méchanique en devenir une nouvelle branche, & me sauront gré d'en avoir étendu ainsi le domaine.*



DES SCIENCES ET BELLES-LETTRES. 173

Il s'enſuit de là que dans le calcul des oſcillations de la mer en vertu de l'attraction du Soleil & de la Lune, on ne peut pas ſuppoſer que la quantité $p\,dx + q\,dy + r\,dz$ ſoit intégrable, puiſqu'elle ne l'eſt pas lorſque le fluide eſt en repos par rapport à la Terre, & qu'il n'a que le mouvement de rotation qui lui eſt commun avec elle.

En général ſi on ſuppoſe p & q fonctions de x & y ſans z ni t, & r conſtante, on aura $\frac{dp}{dt} = 0$, $\frac{dq}{dt} = 0$, $\frac{dr}{dt} = 0$, $\alpha = \frac{dp}{dy} - \frac{dq}{dx}$, $\beta = 0$, $\gamma = 0$; & la quantité qui doit être intégrable (art. 17.) ſera $\left(\frac{dp}{dy} - \frac{dq}{dx}\right)(q\,dx - p\,dy)$. Or par l'incompreſſibilité du fluide on aura $\frac{dp}{dx} + \frac{dq}{dy} = 0$, $\frac{dr}{dx}$ étant nul; donc $p\,dy - q\,dx$ devra être intégrable. Soit donc $p\,dy - q\,dx = d\omega$, on aura $p = \frac{d\omega}{dy}$, $q = -\frac{d\omega}{dx}$, & la quantité $\left(\frac{dp}{dy} - \frac{dq}{dx}\right)(q\,dx - p\,dy)$ deviendra $-\left(\frac{d^2\omega}{dx^2} + \frac{d^2\omega}{dy^2}\right)d\omega$, laquelle devant être elle-même intégrable, il faudra que l'on ait $\frac{d^2\omega}{dx^2} + \frac{d^2\omega}{dy^2} = $ fonct. ω. Ainſi, pourvu que ω ſoit une fonction de x, y, ſans z ni t, laquelle ſatisfaſſe à cette équation, on aura un mouvement poſſible dans le fluide en prenant $p = \frac{d\omega}{dy}$, $q = -\frac{d\omega}{dx}$, $r = $ conſt.; ſans qu'il ſoit néceſſaire que $p\,dx + q\,dy$ ſoit intégrable.

Si on fait $\omega = \frac{g(x^2 + y^2)}{2}$, on aura fonct. $\omega = g$, & $p = gy$, $q = -gx$, comme dans l'exemple précédent.

22. Il y a encore un cas très étendu dans lequel la quantité $p\,dx + q\,dy + r\,dz$ doit être une différentielle exacte. C'eſt celui où l'on ſuppoſe que les viteſſes p, q, r ſoient très petites & qu'on néglige les

Y 3

No figures will be found in this book. The methods here presented require neither constructions, nor geometrical or mechanical reasoning, but only algebraic operations, following a regular and uniform course. Those who love Analysis, will see with pleasure Mechanics become one of its new branches and will be grateful to me for having thus extended its domain.

Helmholtz's Lagrangian vorticity flux invariance (1858)

Tait's 1867 English rendering in the Philosophical Magazine of Helmholtz's 1858 vorticity results for 3D incompressible Euler flow driven by potential forces:

Hermann von Helmholtz 1821-1894

The following investigation shows that when there is a velocity-potential the elements of the fluid have no rotation, but that there is at least a portion of the fluid elements in rotation when there is no velocity-potential.

By *vortex-lines* (*Wirbellinien*) I denote lines drawn through the fluid so as at every point to coincide with the instantaneous axis of rotation of the corresponding fluid element.

By *vortex-filaments* (*Wirbelfäden*) I denote portions of the fluid bounded by vortex-lines drawn through every point of the boundary of an infinitely small closed curve.

The investigation shows that, if all the forces which act on the fluid have a potential,—

1. No element of the fluid which was not originally in rotation is made to rotate.

2. The elements which at any time belong to one vortex-line, however they may be translated, remain on one vortex-line.

3. The product of the section and the angular velocity of an infinitely thin vortex-filament is constant throughout its whole length, and retains the same value during all displacements of the filament. Hence vortex-filaments must either be closed curves, or must have their ends in the bounding surface of the fluid.

THE
LONDON, EDINBURGH, AND DUBLIN
PHILOSOPHICAL MAGAZINE
AND
JOURNAL OF SCIENCE.
SUPPLEMENT TO VOL. XXXIII. FOURTH SERIES.

LXIII. *On Integrals of the Hydrodynamical Equations, which express Vortex-motion. By* H. HELMHOLTZ*.

Translated by Peter Guthrie Tait

If we call *vortex-line* a line whose direction coincides everywhere with the instantaneous axis of rotation of the there-situated element of fluid as above described, we can enunciate the above theorem in the following manner:—*Each vortex-line remains continually composed of the same elements of fluid*, and swims forward with them in the fluid.

The rectangular components of the angular velocity vary directly as the projections of the portion $q\epsilon$ of the axis of rotation; it follows from this that *the magnitude of the resultant angular velocity in a defined element varies in the same proportion as the distance between this and its neighbour along the axis of rotation.*

Conceive that vortex-lines are drawn through every point in the circumference of any indefinitely small surface; there will thus be set apart in the fluid a filament of indefinitely small section which we shall call *vortex-filament*. The volume of a portion of such a filament bounded by two given fluid elements, which (by the preceding propositions) remains filled by the same element of fluid, must in the motion remain constant, and its section must therefore vary inversely as its length. Hence the last theorem may be stated as follows:—*The product of the section and the angular velocity, in a portion of a vortex-filament containing the same element of fluid, remains constant during the motion of that element.*

Clebsch variables (1859)

n dimensions

Alfred Clebsch
1833-1872

- In 3D the velocity 1-form $v := \boldsymbol{v} \cdot d\boldsymbol{x}$ is usually not exact but may be written, following Pfaff and Jacobi, as

$$\boldsymbol{v} \cdot d\boldsymbol{x} = dF + \phi \, d\psi, \qquad \nabla \cdot \boldsymbol{v} = 0$$

which implies for the vorticity vector the Pfaff-Darboux representation $\boldsymbol{\omega} = \nabla\phi \times \nabla\psi$

- Clebsch showed that (the Clebsch variables) ϕ and ψ can be chosen to be material invariants (Lie invariants):

$$(\partial_t + \boldsymbol{v} \cdot \nabla)\phi = 0 \quad (\partial_t + \boldsymbol{v} \cdot \nabla)\psi = 0.$$

- The Clebsch derivation makes use of canonical transformations, taken from Jacobi (1836-1837/1890). In 1861 Hankel found a simple Lagrangian proof (see below).

- But first, we need to take a look at Lagrange's and Cauchy's work on Lagrangian coordinates.

Lagrangian-coordinates formulations

- Lagrange's 1788 formulation of the Euler equations made use of the map $\boldsymbol{a} \mapsto \boldsymbol{x}(\boldsymbol{a},\, t)$ of the initial position $\boldsymbol{a}$ of a fluid particle to its current position $\boldsymbol{x}$, solution of the characteristic equation $\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x},\, t), \quad \boldsymbol{x}(\boldsymbol{a}, 0) = \boldsymbol{a}$. Euler's equations are

$$\ddot{\boldsymbol{x}} = -\nabla p, \qquad \nabla \cdot \dot{\boldsymbol{x}} = 0$$

- By a *pull-back* to (Lagrangian) coordinates, Lagrange obtains:

$$\sum_{k=1}^{3} \ddot{x}_k \nabla^{\mathrm{L}} x_k = -\nabla^{\mathrm{L}} p, \quad \det\left(\nabla^{\mathrm{L}} \boldsymbol{x}\right) = 1$$

where $\nabla^{\mathrm{L}} \boldsymbol{x} := \nabla_{\boldsymbol{a}} \boldsymbol{x}$ is the Jacobian matrix of the map.

- Cauchy (1815) takes the Lagrangian curl of Lagrange's equation which he then integrates in time, to obtain *the Cauchy invariants equations*

$$\sum_{k=1}^{3} \nabla^{\mathrm{L}} \dot{x}_k \times \nabla^{\mathrm{L}} x_k = \nabla^{\mathrm{L}} \times \sum_{k=1}^{3} \dot{x}_k \nabla^{\mathrm{L}} x_k = \boldsymbol{\omega}_0\,,$$

where $\boldsymbol{\omega}_0 = \nabla^{\mathrm{L}} \times \boldsymbol{v}_0$ is the initial vorticity vector.

Hankel’s 1861 Preisschrift

Eur. Phys. J. 2017 **42** 4-5: Frisch-Grimberg-Villone; Villone-Rampf

Hermann Hankel 1839-1873

- In 1860, two years after **Helmholtz** gave his somewhat heuristic derivation of the Lagrangian invariance of the flux of the vorticity through an infinitesimal piece of surface, Göttingen University set up a prize: *The general equations for determining fluids motions may be displayed in two ways, one of which is due to Euler, the other one to Lagrange. The illustrious* ***Dirichlet*** *pointed out in the posthumous paper, titled “On a problem of hydrodynamics'', the hitherto almost totally neglected advantages of the Lagrangian approach; but he seems to have been prevented, by a fatal disease, from a deeper development thereof. So, this Faculty asks for a theory of fluids based on the equations of Lagrange, yielding, at least, the laws of vortex motion discovered otherwise by the illustrious* ***Helmholtz****.*

- Actually, Hankel gave Lagrangian derivations of : Helmholtz’s results, the “Kelvin” circulation theorem, the Clebsch variable representation and the least action formulation for elastic fluids. Riemann: *mancherlei Gutes* (all manner of good things). *Indeed*!

Gott Giebet seinen Kindern auch hier *mancherlei Gutes* im Geistlichen und Leiblichen
Dein Feind is ja auch noch ein Mensch, er hat noch so *mancherlei Gutes* an sich...

Back to Clebsch: Hankel's derivation

- Hankel (1861) uses the Cauchy invariants equations to give a simple "push-forward" derivation of the material (Lie) invariance of the Clebsch variables. He takes an initial vorticity that has a "Pfaff-Darboux" representation:

$$\boldsymbol{\omega}_0 = \nabla^{\mathrm{L}}\phi_0 \times \nabla^{\mathrm{L}}\psi_0 = \nabla^{\mathrm{L}} \times \left(\phi_0 \nabla^{\mathrm{L}}\psi_0\right).$$

- Removing the (Lagrangian) curl - up to a gradient - he gets

$\nabla^{\mathrm{L}} \times \sum_{k=1}^{3} \dot{x}_k \nabla^{\mathrm{L}} x_k = \boldsymbol{\omega}_0$

$$\sum_{k=1}^{3} \dot{x}_k \nabla^{\mathrm{L}} x_k = \nabla^{\mathrm{L}} F^{\mathrm{L}} + \phi_0 \nabla^{\mathrm{L}} \psi_0.$$

- The 2nd term on the rhs is independent of time, whereas the first and the lhs are time-dependent. Hankel performs a push-forward to Eulerian coordinates, obtaining: $\boldsymbol{v} = \nabla F + \phi \nabla \psi,$

  where $F(\boldsymbol{x},t) := F^{\mathrm{L}}(\boldsymbol{a}(\boldsymbol{x},t),t), \quad \phi(\boldsymbol{x},t) := \phi_0(\boldsymbol{a}(\boldsymbol{x},t)), \quad \psi(\boldsymbol{x},t) := \psi_0(\boldsymbol{a}(\boldsymbol{x},t))$

  and $\boldsymbol{a}(\boldsymbol{x},t)$ is the inverse of the Lagrangian map. Obviously ϕ and ψ remain constant along fluid particle trajectory. Thus

$$\partial_t \phi + \boldsymbol{v} \cdot \nabla \phi = 0 \text{ and } \partial_t \psi + \boldsymbol{v} \cdot \nabla \psi = 0.$$

Hankel's proof of the Helmholtz and circulation theorems

Hankel uses Cauchy's 1815 invariants equations, in the form

$$\nabla^{\mathrm{L}} \times \sum_{k=1}^{3} \dot{x}_k \nabla^{\mathrm{L}} x_k = \boldsymbol{\omega}_0$$

In the Lagrangian space of initial fluid positions he takes a finite piece of smooth surface S_0 limited by a contour C_0 and their images by the Lagrangian map from 0 to *t*, S and C, respectively.

He then applies the *Kelvin*-Stokes-*Hankel* theorem at time zero and at time *t*, to obtain

circulation

$$\int_{C_0} \boldsymbol{v}_0 \cdot d\boldsymbol{a} = \int_{S_0} \boldsymbol{\omega}_0 \cdot \boldsymbol{n}_0 \, d\sigma_0 = \int_{C_0} \Sigma_k v_k \nabla^{\mathrm{L}} x_k \cdot d\boldsymbol{a} = \int_{C} \boldsymbol{v} \cdot d\boldsymbol{x} = \int_{S} \boldsymbol{\omega} \cdot \boldsymbol{n} \, d\sigma$$

vorticity flux

where $\boldsymbol{\omega} := \nabla \times \boldsymbol{v}$ and use has been made of $\sum_{k=1}^{3} v_k \nabla^{\mathrm{L}} x_k \cdot d\boldsymbol{a} = \boldsymbol{v} \cdot d\boldsymbol{x}$

Hankel has thus not only proved Helmholtz's theorem, but obtained an *integral invariant* (circulation theorem), which states that:

$$\int_{C_0} \boldsymbol{v}_0 \cdot d\boldsymbol{a} = \int_{C} \boldsymbol{v} \cdot d\boldsymbol{x} \, .$$

Integral invariants in (non-)relativistic fluid dynamics

- In *Nouvelles Méthodes de la Mécanique Céleste,* vol. III (Integral invariants), Poincaré assigns the circulation theorem, not to Kelvin (whose derivation was widely known) nor to Hankel (whose Preisschrift book was unknown in France), but to Helmholtz. In a sense he was right: by "Stokes's" theorem, the circulation along a closed curve equals the vorticity flux through a surface bordered by that curve. Since Helmholtz proved the Lagrangian invariance of vorticity flux through infinitesimal surfaces, the circulation invariance follows by additivity.

William Thomson (Kelvin) 1824-1907

John Lighton Synge 1897-1995

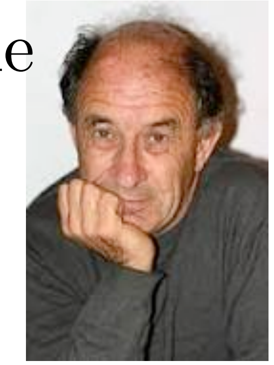
André Lichnerowicz 1915-1998

- Still, Hankel can be credited for making the transition from a differential (Lie) invariant to a global (Poincaré-Cartan) invariant, using the "Stokes" theorem and Cauchy's Lagrangian formulation of the Euler eqations.
- About twenty years after Einstein's introduction of GR, Synge extended Helmholtz's results to hydrodynamics in a GR background and Lichnerowicz gave the interpretation, using Elie Cartan's integral invariants, a time-dependent extension of Poincaré's integral invariants, much better adapted to GR. The distinction between Helmholtz's kinematic and dynamic invariants becomes then blurred.

The infinite-dimensional geometrization of the Lagrangian approach: Arnold (1966)

Arnold (1966) (Ann. Inst. Fourier): The solutions of the incompressible Euler equations extremize the action :

Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits

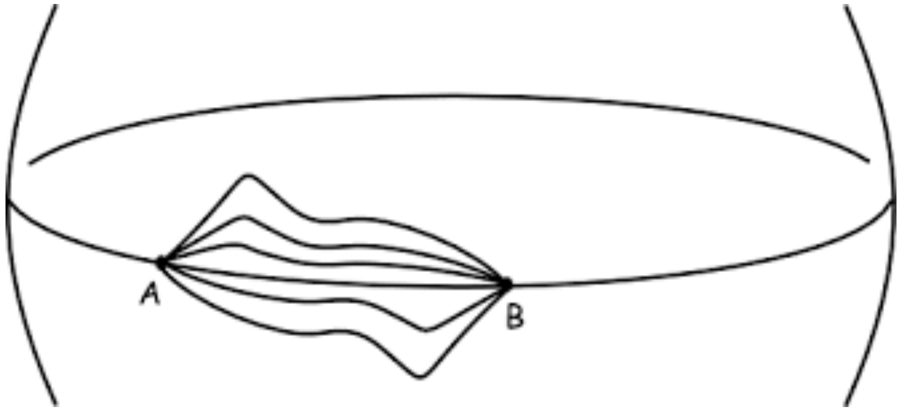


Vladimir Arnold
1937 - 2010

$$A = \int_0^T dt \int d^3a \, \frac{1}{2} |\partial \boldsymbol{x}(\boldsymbol{a}, t)/\partial t|^2,$$

with the constraints $J = 1$, $\boldsymbol{x}(\boldsymbol{a}, 0) = \boldsymbol{a}$ and given $\boldsymbol{x}(\boldsymbol{a}, T)$.
In geometrical language, they are g*eodesics of SDiff*.

An elementary example of geodesic on the sphere



Conclusion. The geometry that **Lagrange** rejected in 1788 was that of the Greeks. He loved playing with differential forms, which for him had no geometrical content. Thanks to Helmholtz, Riemann, Hankel, Poincaré, Cartan, Arnold and many more, we now realise that fluid dynamicists never stopped doing geometry.