Long-Alfvén-wave trains in collisionless plasmas. I. Kinetic theory

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A generalized kinetic derivative nonlinear Schrödinger equation for the multidimensional dynamics of Alfvén wave trains propagating along an ambient magnetic field is derived from the Vlasov-Maxwell equations by a reductive perturbative expansion. It retains in addition to the Landau damping, the coupling to longitudinally averaged fields driven by both transverse gradients and kinetic effects. These mean fields that modulate the propagation speed of the wave play a main role in transverse instabilities of extended Alfvén wave packets and in the filamentation phenomenon. This long-wave model also provides a benchmark for Landau-fluid descriptions of collisionless plasmas.

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I. INTRODUCTION

Alfvén waves propagating along an ambient magnetic field are ubiquitous in space plasmas and play an important role in the physics of media such as the solar corona, the solar wind or the terrestrial magnetosheath. Their dynamics is often addressed in the framework of the magnetohydrodynamics (MHD) or more generally of the Hall-MHD when the fluctuation spectrum extends up to scales comparable to the ion gyromagnetic radius, making relevant the dispersive effect due to ion inertia. Collisions are however essentially negligible in such media, and the usual fluid description is thus questionable. On the other hand, direct numerical integrations of the Vlasov-Maxwell equations in three space dimensions are in most situations much beyond the capabilities of the present day computers. This suggests the development of a reduced description that, while retaining most of the aspects of a fluid description, incorporates a realistic approximation of the Landau damping. Such models consist in a system of moment equations for mass densities, momenta and gyrotropic pressure components. These equations involve heat fluxes, making the resulting system unclosed. Additional relations between the fields are then heuristically supplemented, in a way that reproduces the linear Landau damping. This approach, developed in the MHD regime by Snyder, Hammett and Dorland,[1] is revisited in the companion paper (paper II).[2]

Closed asymptotic models for collisionless plasmas can however be derived in specific instances. Such an approach was initiated three decades ago by Rogister[3] who applied a reductive perturbative expansion to the Vlasov-Maxwell equations to isolate the dynamics of small-amplitude Alfvén waves with a typical length much larger than the ion Larmor radius. This analysis led to a long-wave equation that is especially simple in one space dimension and was later named "kinetic derivative nonlinear Schrödinger equation" (KDNLS) to refer to the linear Landau damping retained by this asymptotics. This kinetic effect which originates from the resonant interactions with the particles leads to a nonlocal contribution involving a Hilbert transform in the nonlinear terms of the equation. This long-wave model was reproduced by Mjølhus and Wyller[4] who used a mixed approach where a reductive perturbative expansion is performed in the framework of Hall-MHD with finite Larmor radius corrections,[5] but where the gyrotropic components of the pressure are evaluated from the guiding center distribution function. An approach based on fluid moments with Landau damping modeled by additional dissipation-like terms was also proposed.[6] Nonlinear Landau damping and particle trapping in finite amplitude waves were also recently considered.[7]

In the case of quasi-monochromatic wave trains, the coupling to large-scale magnetosonic waves must be retained, as first shown in the Hall-MHD context.[8,9] This coupling involves mean values along the direction of propagation of the density and of the longitudinal velocity and magnetic field components. They vanish in the case of localized solutions but were shown to play an essential role in the phenomenon of transverse collapse of a sufficiently extended quasi-monochromatic Alfvén wave packet,[10,11] resulting in the formation of intense magnetic filaments.[12,13] It was in particular established that in the long-wave limit absolute filamentation takes place when the β of the plasma is larger than unity. In such warm plasmas, kinetic effects are however important. It is thus necessary to develop a long-wave formalism that enables ones to address the Alfvén wave filamentation problem in a realistic setting.[14] The multidimensional KDNLS equations can in addition provide a tool to benchmark Landau-fluid models in regimes going beyond linear instabilities.

In the present paper, a KDNLS equation valid in the case of multidimensional wave trains is derived by revisiting Rogister's reductive perturbative expansion,[3] avoiding in particular the Fourier mode decomposition. The expansion is pushed to an order high enough to include the coupling to the mean fields driven by both transverse gradients and kinetic effects. In the companion paper,[2] the present analysis is used as the starting point for the construction and validation of a Landau-fluid model including Hall-effect that is expected to provide an efficient tool for realistic simulations of dispersive Alfvén wave turbulence.

II. THE DYNAMICS OF LONG ALFVÉN WAVES

A. The reductive perturbative expansion

The dynamics of Alfvén waves propagating along a strong ambient magnetic field are amenable to an asymptotic expansion, directly from the Vlasov-Maxwell equation, when involving scales that are large compared to the ion Larmor radius and amplitudes small enough to keep linear dispersive effects comparable to the nonlinearities.[3] For this purpose, one writes the Vlasov-Maxwell equations in the form

$$\partial_t f_r + v\cdot\nabla f_r + \frac{q_r}{m_r}(e+\frac{1}{c}v\times b)\cdot\nabla_v f_r = 0 \tag{1}$$
$$\frac{1}{c}\partial_t b = -\nabla\times e \tag{2}$$
$$\nabla\times b = \frac{4\pi}{c}\sum_r q_r n_r\int v f_r d^3v + \frac{1}{c}\partial_t e \tag{3}$$
$$\nabla\cdot e = 4\pi\sum_r q_r n_r\int f_r d^3v, \tag{4}$$

where f_r and n_r are the distribution function and the average number density of the particle species r with charge q_r and mass m_r. As checked below, local neutrality of the plasma holds at all the relevant orders and the displacement current $\frac{1}{c}\partial_t e$ turns out to be negligible in the present analysis. This contribution which might be important for auroral plasmas is retained by Verheest.[15]

The reductive perturbation expansion proceeds like in the derivation of the derivative nonlinear Schrödinger (DNLS) equation from the Hall-MHD equations.[4,8,12] For an ambient field $B_0\widehat{x}$ (where $\widehat{x}$ is the unit vector pointing in the x-direction), one expands the distribution function and the electric and magnetic fields in the form $f_r = F_r^{(0)} + \epsilon(f_r^{(0)} + \epsilon f_r^{(1)} + \ldots)$, $b = B_0\widehat{x} + \epsilon(b^{(0)} + \epsilon b^{(1)} + \ldots)$ and $e = \epsilon(e^{(0)} + \epsilon e^{(1)} + \ldots)$, where $F_r^{(0)}$ denotes the equilibrium velocity distribution function, assumed rotationally symmetric around the direction of the ambient field and symmetric relatively to forward and backward velocities along this direction, thus excluding the presence of equilibrium drifts.[15–17] Denoting by λ the Alfvén-wave propagation velocity that will be determined in the following, one defines the stretched coordinates $\xi = \epsilon^2(x-\lambda t)$, $\eta = \epsilon^3 y$ and $\zeta = \epsilon^3 z$ and the slow time $\tau = \epsilon^4 t$. It is also convenient to express the velocity v in a cylindrical coordinate system by defining the velocity space angle $\phi = \tan^{-1}(v_z/v_y)$. One writes $v = (v_\parallel\ ,\ v_\perp\cos\phi\ ,\ v_\perp\sin\phi) = (v_\parallel, v_\perp\vec{n}) = (v_\parallel, \vec{v}_\perp$ and

$$\nabla_v = (\partial_{v_\parallel}\ ,\ \cos\phi\,\partial_{v_\perp} - \frac{\sin\phi}{v_\perp}\partial_\phi\ ,\ \sin\phi\,\partial_{v_\perp} + \frac{\cos\phi}{v_\perp}\partial_\phi) = (\partial_{v_\parallel}\ ,\ \vec{n}\,\partial_{v_\perp} + \vec{t}\,\frac{1}{v_\perp}\partial_\phi), \tag{5}$$

where $\vec{n} = (\cos\phi, \sin\phi)$ and $\vec{t} = \frac{d\vec{n}}{d\phi}$. Furthermore, $\frac{q_r}{cm_r}(v\times B_0\widehat{x})\cdot\nabla_v = -\Omega_r\partial_\phi$, where $\Omega_r = \frac{q_r B_0}{m_r c}$ is the cyclotron frequency of the particles of species r. It is also useful to note that $\partial_\phi\nabla_v = \nabla_v\partial_\phi + \widehat{x}\times\nabla_v$. Expanding to the successive orders, one gets from eq. (1),

$$\Omega_r\partial_\phi F_r^{(0)} = 0 \tag{6}$$
$$\Omega_r\partial_\phi f_r^{(0)} = \Sigma_r^{(0)} \tag{7}$$
$$\Omega_r\partial_\phi f_r^{(1)} = \Sigma_r^{(1)} \tag{8}$$
$$\Omega_r\partial_\phi f_r^{(2)} = (v_\parallel-\lambda)\partial_\xi f_r^{(0)} + \Sigma_r^{(2)} \tag{9}$$
$$\Omega_r\partial_\phi f_r^{(3)} = (v_\parallel-\lambda)\partial_\xi f_r^{(1)} + (\vec{v}_\perp\cdot\nabla_\perp)f_r^{(0)} + \Sigma_r^{(3)} \tag{10}$$
$$\Omega_r\partial_\phi f_r^{(4)} = (v_\parallel-\lambda)\partial_\xi f_r^{(2)} + (\vec{v}_\perp\cdot\nabla_\perp)f_r^{(1)} + \partial_\tau f_r^{(0)} + \Sigma_r^{(4)} \tag{11}$$
$$\Omega_r\partial_\phi f_r^{(5)} = (v_\parallel-\lambda)\partial_\xi f_r^{(3)} + (\vec{v}_\perp\cdot\nabla_\perp)f_r^{(2)} + \partial_\tau f_r^{(1)} + \Sigma_r^{(5)} \tag{12}$$

where

$$\Sigma_r^{(s)} = \frac{q_r}{m_r}\left(e^{(s)} + \frac{1}{c}v\times b^{(s)}\right)\cdot\nabla_v F_r^{(0)} + \sum_{p+q=s-1}\frac{q_r}{m_r}\left(e^{(p)} + \frac{1}{c}v\times b^{(p)}\right)\cdot\nabla_v f_r^{(q)}, \tag{13}$$

the second term of the right-hand-side (RHS) being absent when $s=0$.

From eq. (2), one derives

$$\widehat{x}\times\partial_\xi e^{(0)} = \frac{\lambda}{c}\partial_\xi b^{(0)} \tag{14}$$

$$\widehat{x}\times\partial_\xi e^{(1)} + \nabla_\perp\times e^{(0)} = \frac{\lambda}{c}\partial_\xi b^{(1)} \tag{15}$$

$$\widehat{x}\times\partial_\xi e^{(2)} + \nabla_\perp\times e^{(1)} = \frac{\lambda}{c}\partial_\xi b^{(2)} - \frac{1}{c}\partial_\tau b^{(0)}. \tag{16}$$

and more generally for any $p\geq 0$,

$$\widehat{x}\times\partial_\xi e^{(2+p)} + \nabla_\perp\times e^{(1+p)} = \frac{\lambda}{c}\partial_\xi b^{(2+p)} - \frac{1}{c}\partial_\tau b^{(p)}. \tag{17}$$

It immediately follows from eq. (14) that $b_\parallel^{(0)}=0$ and $e_\perp^{(0)} = -\frac{\lambda}{c}\widehat{x}\times b_\perp^{(0)}$, where a zero mean value is assumed for the transverse magnetic field $b_\perp^{(0)}$ when averaged over the ξ variable. Such a transverse mean field that when present obeys the reduced MHD equations, is not created dynamically in the long-wave asymptotics that does not retain the interactions between counter propagating waves.[9,18]

When averaged on the angle ϕ, eq. (7) gives $e_\parallel^{(0)}=0$, and thus rewrites

$$\partial_\phi f_r^{(0)} = \frac{1}{B_0}\left(\partial_\phi(b_\perp^{(0)}\cdot\vec{v}_\perp)\partial_{v_\parallel}F_r^{(0)} - (v_\parallel-\lambda)b_\perp^{(0)}\cdot\partial_\phi\nabla_{v_\perp}F_r^{(0)}\right). \tag{18}$$

Defining for any positive integer j the operator

$$\mathcal{D}_j^{\pm} = (\lambda - v_\parallel)\left(\frac{\partial}{\partial v_\perp} \pm \frac{j}{v_\perp}\right) + v_\perp\frac{\partial}{\partial v_\parallel}, \tag{19}$$

and also $D\equiv\mathcal{D}_0^{\pm}$, one writes

$$f_r^{(0)} = \frac{1}{B_0}(b_\perp^{(0)}\cdot\vec{n})\mathcal{D}F_r^{(0)} + \overline{f}_r^{(0)}, \tag{20}$$

where the last term of the RHS refers to a possible contribution independent of the ϕ variable, that will be later shown to be identically zero.

The solvability condition for eq. (8) reads $e_\parallel^{(1)}=0$. The parallel component of eq. (15) reproduces the divergenceless condition

$$\partial_\xi b_\parallel^{(1)} + \nabla_\perp\cdot b_\perp^{(0)} = 0, \tag{21}$$

while the transverse component gives $e_\perp^{(1)} = -\frac{\lambda}{c}\widehat{x}\times b_\perp^{(1)}$. Since this relation is the same as that relating $e_\perp^{(0)}$ and $b_\perp^{(0)}$, one can prescribe $e_\perp^{(1)}=0$ and $b_\perp^{(1)}=0$. In particular, the modulus of the local magnetic field is given by $|b| = [(B_0+\epsilon^2 b_\parallel^{(1)})^2 + \epsilon^2|b_\perp|^2+\cdots]^{\frac{1}{2}} = B_0(1+\epsilon^2 A) + O(\epsilon^3)$ with $A = \frac{b_\parallel^{(1)}}{B_0} + \frac{|b_\perp^{(0)}|^2}{2B_0^2}$. Similarly, (17) implies that for $p>0$, one has $e_\perp^{(2p+1)}=0$, $b_\perp^{(2p+1)}=0$, $e_\parallel^{(2p+2)}=0$ and $b_\parallel^{(2p+2)}=0$. It follows that $\partial_\phi f_r^{(1)} = \frac{c}{B_0}\left(e_\perp^{(0)} + \frac{v\times b^{(0)}}{c}\right)\cdot\nabla_v f_r^{(0)}$ or

$$\partial_\phi f_r^{(1)} = \frac{1}{B_0}[(v_\parallel-\lambda)(\widehat{x}\times b_\perp^{(0)})\cdot\nabla_{v_\perp}f_r^{(0)} + (\vec{v}_\perp\times b_\perp^{(0)})\partial_{v_\parallel}f_r^{(0)}]. \tag{22}$$

The solvability condition for eq. (9) reads

$$(v_\parallel - \lambda)\partial_\xi \overline{f}_r^{(0)} + \frac{q_r}{m_r} e_\parallel^{(2)} \partial_{v_\parallel} F_r^{(0)} + \sum_{p+q=1} \frac{q_r}{m_r} \langle (e^{(p)} + \frac{1}{c} v \times b^{(p)}) \cdot \nabla_v f_r^{(q)} \rangle = 0, \tag{23}$$

where $\langle \cdot \rangle = \dfrac{1}{2\pi} \displaystyle\int \cdot \, d\phi$ denotes the averaging on the ϕ variable. One easily checks that in the sum, the term associated with the indices $p = 1$, $q = 0$ does not contribute. The other term is estimated by noticing that $\langle \nabla_{v_\perp} f_r^{(1)} \rangle = \left(\partial_{v_\perp} + v_\perp^{-1}\right) \langle \partial_\phi f_r^{(1)} \vec{t} \rangle$, and, using the explicit form of $\partial_\phi f_r^{(1)}$, one easily checks that the only possible contribution comes from $\overline{f}_r^{(0)}$. The solvability condition takes the form

$$(v_\parallel - \lambda)\partial_\xi \overline{f}_r^{(0)} + \frac{q_r}{m_r} e_\parallel^{(2)} \partial_{v_\parallel} F_r^{(0)} + L\overline{f}_r^{(0)} = 0, \tag{24}$$

where L is a linear operator that it is not necessary to specify. To estimate $e^{(2)}$, one considers eq. (16) that, using the condition $e^{(1)} = 0$, reduces to $e_\perp^{(2)} = -\frac{\lambda}{c} \widehat{x} \times b_\perp^{(2)} + \frac{1}{c} \widehat{x} \times \partial_\xi^{-1} \partial_\tau b_\perp^{(0)}$ where ∂_ξ^{-1} denotes the anti-derivative. One thus consistently solves as $e_\parallel^{(2)} = 0$ and $b_\parallel^{(2)} = 0$. It follows that, as announced, $\overline{f}_r^{(0)} = 0$.

B. Propagation velocity of the wave

Applying the vectorial operator $\sum_r m_r n_r \int d^3 v \, \vec{v}_\perp$ on the two sides of the eq. (9) allows one to determine the propagation velocity λ of the Alfvén wave. In the left-hand-side (LHS) of the resulting equation, one uses eq. (A.3) and gets

$$\sum_r n_r m_r \Omega_r \int \vec{v}_\perp \partial_\phi f_r^{(2)} d^3 v = \frac{B_0}{4\pi} \partial_\xi b_\perp^{(0)}. \tag{25}$$

In the RHS, one has

$$\begin{aligned} \int (v_\parallel - \lambda) \vec{v}_\perp \partial_\xi f_r^{(0)} d^3 v &= \frac{1}{B_0} \int (v_\parallel - \lambda) \vec{v}_\perp (\partial_\xi b_\perp^{(0)} \cdot \vec{n}) \mathcal{D} F_r^{(0)} d^3 v \\ &= \frac{1}{B_0} \partial_\xi b_\perp^{(0)} \Big[\int (v_\parallel - \lambda)^2 F_r^{(0)} d^3 v - \frac{1}{2} \int v_\perp^2 F_r^{(0)} d^3 v \Big]. \end{aligned} \tag{26}$$

This yields

$$\sum_r m_r n_r \int (v_\parallel - \lambda) \vec{v}_\perp \partial_\xi f_r^{(0)} d^3 v = \frac{1}{B_0} (p_\parallel^{(0)} - p_\perp^{(0)} + \lambda^2 \rho^{(0)}) \, \partial_\xi b_\perp^{(0)} \tag{27}$$

that involves the equilibrium parallel and transverse pressures $p_\parallel^{(0)} = \sum_r m_r n_r \int v_\parallel^2 F_r^{(0)} d^3 v$ and $p_\perp^{(0)} = \displaystyle\sum_r m_r n_r \int \frac{v_\perp^2}{2} F_r^{(0)} d^3 v$, together with the corresponding density $\rho^{(0)} = \sum_r m_r n_r \int F_r^{(0)} d^3 v$. In order to estimate the contribution from $\Sigma_r^{(2)}$, one writes

$$\sum_r n_r q_r \int \vec{v}_\perp (e_\perp^{(2)} + \frac{v \times b_\perp^{(2)}}{c}) \cdot \nabla_v F_r^{(0)} d^3 v = -\sum_r q_r e_\perp^{(2)}, \tag{28}$$

which vanishes as the result of the neutrality condition $\sum_r q_r n_r = 0$ (see Appendix A). Using again the neutrality condition one has

$$\begin{aligned} &\sum_{p+q=1} \sum_r n_r q_r \int \vec{v}_\perp (e_\perp^{(p)} + \frac{v \times b^{(p)}}{c}) \cdot \nabla_v f_r^{(q)} d^3 v = \\ &\qquad - \sum_r n_r q_r \int \Big(\frac{v_\parallel \times b_\perp^{(0)}}{c} f_r^{(1)} + \frac{\vec{v}_\perp \times b_\parallel^{(1)}}{c} f_r^{(0)} \Big) d^3 v = 0. \end{aligned} \tag{29}$$

To simplify the writing, the same notation is used here and in the following to denote a vector pointing along the ambient field and its unique component (e.g. $v_\parallel \equiv v_\parallel \widehat{x}$ and $b_\perp^{(0)} \times b_\perp^{(2)} \equiv (b_\perp^{(0)} \times b_\perp^{(2)}) \cdot \widehat{x}$).

One concludes that the propagation velocity λ is given by

$$\lambda^2 \rho^{(0)} = \frac{|B_0|^2}{4\pi} + p_\perp^{(0)} - p_\parallel^{(0)}, \tag{30}$$

where the usual Alfvén velocity is affected by the anisotropy of the equilibrium pressure tensor. In order to prevent the system to be firehose unstable, the RHS of eq. (30) is assumed positive.

C. Wave-particle resonance

The solvability condition for eq. (10) together with the quasi-neutrality constraint $\sum_r q_r n_r \int \overline{f}_r^{(1)} d^3v = 0$ (see Appendix A) prescribe $\overline{f}_r^{(1)}$ and $e_\parallel^{(3)}$. One has

$$(v_\parallel - \lambda)\partial_\xi \overline{f}_r^{(1)} = -\langle (\vec{v}_\perp \cdot \nabla_\perp) f_r^{(0)} \rangle - \frac{q_r}{m_r}\frac{\partial F_r^{(0)}}{\partial v_\parallel} e_\parallel^{(3)} - \frac{q_r}{m_r}\sum_{p+q=2} \langle \Big(e^{(p)} + \frac{v \times b^{(p)}}{c}\Big) \cdot \nabla_v f_r^{(q)} \rangle \tag{31}$$

that displays a singularity when the longitudinal velocity $v_\parallel$ of the particles equals the propagation velocity λ of the wave. Near this resonance, the time derivative $\epsilon^2 \partial_\tau \overline{f}_r^{(1)}$ is no more subdominant and affects the estimate of the integral $\int \partial_\xi \overline{f}_r^{(1)} dv_\parallel$. We will return to this point after estimating the various terms entering the RHS of eq. (31).

Using eqs. (B.1)-(B.2) and the divergenceless condition (21), one easily gets

$$\langle (\vec{v}_\perp \cdot \nabla_\perp) f_r^{(0)} \rangle = -\frac{1}{2B_0}\partial_\xi b_\parallel^{(1)} v_\perp \mathcal{D} F_r^{(0)}. \tag{32}$$

Separating the longitudinal and transverse contributions of the velocity-gradient, one writes

$$\begin{aligned}\sum_{p+q=2} \Big\langle (e^{(p)} + \frac{v \times b^{(p)}}{c}) \cdot \nabla_v f_r^{(q)} \Big\rangle = \Big\langle & \frac{v_\parallel - \lambda}{c}(\widehat{x} \times b_\perp^{(0)}) \cdot \nabla_{v_\perp} f_r^{(2)} \\ & - \frac{1}{c}(\vec{v}_\perp \cdot b_\perp^{(0)}) \frac{\partial}{\partial v_\parallel}\frac{\partial f_r^{(2)}}{\partial \phi} + \frac{v_\parallel - \lambda}{c}(\widehat{x} \times b_\perp^{(2)}) \cdot \nabla_{v_\perp} f_r^{(0)} \\ & + \frac{1}{c}\partial_\xi^{-1}\partial_\tau (\widehat{x} \times b_\perp^{(0)}) \cdot \nabla_{v_\perp} f_r^{(0)} - \frac{\vec{v}_\perp \cdot b_\perp^{(2)}}{c}\frac{\partial}{\partial v_\parallel}\frac{\partial f_r^{(0)}}{\partial \phi}\Big\rangle. \end{aligned} \tag{33}$$

Using the equalities

$$\langle (\widehat{x} \times b_\perp^{(p)}) \cdot \nabla_{v_\perp} f^{(q)} \rangle = \Big\langle (b_\perp^{(p)} \cdot \vec{v}_\perp)\frac{1}{v_\perp}\left(\frac{\partial}{\partial v_\perp} + \frac{1}{v_\perp}\right)\frac{\partial f^{(q)}}{\partial \phi}\Big\rangle, \tag{34}$$

and

$$\begin{aligned}\langle \partial_\xi^{-1}\partial_\tau (\widehat{x} \times b_\perp^{(0)}) \cdot \nabla_{v_\perp} f_r^{(0)} \rangle = {} & \partial_\xi^{-1}\partial_\tau b_y^{(0)} \Big\langle \cos\phi \frac{\partial}{\partial v_\perp}\frac{\partial f_r^{(0)}}{\partial \phi} + \cos\phi \frac{1}{v_\perp}\frac{\partial f_r^{(0)}}{\partial \phi}\Big\rangle \\ & + \partial_\xi^{-1}\partial_\tau b_z^{(0)} \Big\langle \sin\phi \frac{\partial}{\partial v_\perp}\frac{\partial f_r^{(0)}}{\partial \phi} + \sin\phi \frac{1}{v_\perp}\frac{\partial f_r^{(0)}}{\partial \phi}\Big\rangle, \end{aligned} \tag{35}$$

together with the explicit expression (20) of $f_r^{(0)}$, one gets

$$\begin{aligned}\sum_{p+q=2} \Big\langle (e^{(p)} + \frac{v \times b^{(p)}}{c}) \cdot \nabla_v f_r^{(q)} \Big\rangle = {} & -\frac{1}{c}\mathcal{D}_1^+ \Big(b_y^{(0)} \langle \cos\phi \frac{\partial}{\partial \phi} f_r^{(2)} \rangle + b_z^{(0)} \langle \sin\phi \frac{\partial}{\partial \phi} f_r^{(2)} \rangle \\ & + \frac{1}{2B_0}\mathcal{D} F_r^{(0)} (b_\perp^{(2)} \times b_\perp^{(0)})\Big) + \frac{1}{2B_0 c}(\partial_\xi^{-1}\partial_\tau b_\perp^{(0)} \times b_\perp^{(0)})\left(\frac{\partial}{\partial v_\perp} + \frac{1}{v_\perp}\right)\mathcal{D} F_r^{(0)}. \end{aligned} \tag{36}$$

In the first term of the RHS one uses eqs. (B.5) and (B.6) to write

$$b_y^{(0)}\langle\cos\phi\,\frac{\partial}{\partial\phi}f_r^{(2)}\rangle+b_z^{(0)}\langle\sin\phi\,\frac{\partial}{\partial\phi}f_r^{(2)}\rangle=-\frac{m_rc}{4q_rB_0^2}\partial_\xi|b_\perp^{(0)}|^2(\lambda-v_\|)\mathcal{D}F_r^{(0)}$$
$$+\frac{1}{2B_0}(\partial_\xi^{-1}\partial_\tau b_\perp^{(0)}\times b_\perp^{(0)})\frac{\partial F_r^{(0)}}{\partial v_\perp}+\frac{1}{2B_0}(b_\perp^{(0)}\times b_\perp^{(2)})\mathcal{D}F_r^{(0)}. \tag{37}$$

This leads to

$$(v_\|-\lambda)\partial_\xi\overline{f}_r^{(1)}=\frac{1}{2B_0}\partial_\xi b_\|^{(1)}v_\perp\mathcal{D}F_r^{(0)}-\frac{q_r}{m_r}e_\|^{(3)}\frac{\partial F_r^{(0)}}{\partial v_\|}$$
$$+\frac{1}{4B_0^2}\partial_\xi|b_\perp^{(0)}|^2\mathcal{D}_1^+(v_\|-\lambda)\mathcal{D}F_r^{(0)}-\frac{1}{cB_0}\frac{q_r}{m_r}(\partial_\xi^{-1}\partial_\tau b_\perp^{(0)}\times b_\perp^{(0)})\frac{\partial F_r^{(0)}}{\partial v_\|}, \tag{38}$$

that is conveniently rewritten as

$$(v_\|-\lambda)\partial_\xi\overline{f}_r^{(1)}=(v_\|-\lambda)R_r+S_r\frac{\partial F_r^{(0)}}{\partial v_\|} \tag{39}$$

with

$$R_r=-\frac{v_\perp}{2}\frac{\partial F_r^{(0)}}{\partial v_\perp}\partial_\xi A+\frac{1}{2}\mathcal{D}_1^+\mathcal{D}F_r^{(0)}\partial_\xi\frac{|b_\perp^{(0)}|^2}{2B_0^2} \tag{40}$$

$$S_r=\frac{v_\perp^2}{2}\partial_\xi A+\frac{q_r}{m_r}\partial_\xi\varphi \tag{41}$$

where

$$-\partial_\xi\varphi=e_\|^{(3)}+\frac{1}{cB_0}\partial_\xi^{-1}\partial_\tau b_\perp^{(0)}\times b_\perp^{(0)} \tag{42}$$

identifies with the electric field along the local magnetic field. One thus gets $\partial_\xi\overline{f}_r^{(1)}=R_r+\chi_r$, where χ_r corresponds to the singular contribution driven by $S_r\frac{\partial F_r^{(0)}}{\partial v_\|}$. This singularity results from the assumption that in the frame traveling at the propagation velocity λ of the wave, all the dynamics takes place on a time scale $O(\epsilon^{-4})$, a condition that is broken near the resonance. One should thus define[4] $\chi_r=\lim_{\epsilon\to 0}\chi_r^{\{\epsilon\}}$ where $\chi_r^{\{\epsilon\}}$ obeys

$$\epsilon^2\partial_\tau\chi_r^{\{\epsilon\}}+(v_\|-\lambda)\partial_\xi\chi_r^{\{\epsilon\}}=\partial_\xi S_r\frac{\partial F_r^{(0)}}{\partial v_\|}. \tag{43}$$

Since S_r does not depend on $v_\|$, one can explicitly solve in the form

$$\chi_r^{\{\epsilon\}}(\xi,\tau)=\chi_r^{\{\epsilon\}}(\xi-(v_\|-\lambda)\frac{\tau}{\epsilon^2},0)+\frac{1}{v_\|-\lambda}\frac{\partial F_r^{(0)}}{\partial v_\|}\Big(S_r(\xi)-S_r(\xi-(v_\|-\lambda)\frac{\tau}{\epsilon^2})\Big). \tag{44}$$

The condition $\sum_r q_rn_r\int\overline{f}_r^{(1)}d^3v=0$ then implies

$$\sum_r q_rn_r\int R_rd^3v+2\pi\sum_r q_rn_r\int d(\frac{v_\perp^2}{2})\int\chi_r\,dv_\|=0. \tag{45}$$

One easily checks that, as a consequence of the electric neutrality of the equilibrium state, $\sum_r q_rn_r\int R_rd^3v=0$. One then computes

$$\int\chi_r\,dv_\|=\lim_{\epsilon\to 0}\int\chi_r^{\{\epsilon\}}\,dv_\|=\mathrm{P}\int\frac{1}{v_\|-\lambda}\frac{\partial F_r^{(0)}}{\partial v_\|}\,dv_\|\,S_r(\xi)+\pi\frac{\partial F_r^{(0)}}{\partial v_\|}\Big|_{v_\|=\lambda}\mathcal{H}_\xi\{S_r\} \tag{46}$$

where P holds for principal value and $\mathcal{H}_\xi\{S_r\} = \frac{1}{\pi}\mathrm{P}\int \frac{S_r(\zeta)}{\zeta-\xi}d\zeta$ denotes the Hilbert transform with respect to the ξ variable. It is convenient to write $\int \chi_r \, dv_\parallel = \mathcal{G}_r S_r$, with

$$\mathcal{G}_r = \mathrm{P}\int \frac{1}{v_\parallel - \lambda}\frac{\partial F_r^{(0)}}{\partial v_\parallel}\, dv_\parallel + \pi \frac{\partial F_r^{(0)}}{\partial v_\parallel}\Big|_{v_\parallel=\lambda}\mathcal{H}_\xi. \tag{47}$$

It follows that

$$2\pi\sum_r q_r n_r \Big[\int d(\frac{v_\perp^2}{2})\frac{v_\perp^2}{2}\mathcal{G}_r\partial_\xi A + \frac{q_r}{m_r}\int d(\frac{v_\perp^2}{2})\mathcal{G}_r\partial_\xi\varphi\Big] = 0, \tag{48}$$

or equivalently

$$-\partial_\xi\varphi = \mathcal{L}^{-1}\mathcal{M}\,\partial_\xi A, \tag{49}$$

where one has defined the operators[4]

$$\mathcal{L} = 2\pi\sum_r \frac{q_r^2 n_r}{m_r}\int_0^\infty d(\frac{v_\perp^2}{2})\,\mathcal{G}_r \quad , \quad \mathcal{M} = 2\pi\sum_r q_r n_r \int_0^\infty d(\frac{v_\perp^2}{2})\frac{v_\perp^2}{2}\,\mathcal{G}_r. \tag{50}$$

It follows that eq. (41) simplifies into

$$S_r = \Big(\frac{v_\perp^2}{2} - \frac{q_r}{m_r}\mathcal{L}^{-1}\mathcal{M}\Big)\partial_\xi A. \tag{51}$$

Note that the inversion of the operator $\mathcal{L}$ is easily performed due to the property $\mathcal{H}_\xi = -\mathcal{H}_\xi^{-1}$ of the Hilbert transform. It follows that

$$\int \partial_\xi \overline{f}_r^{(1)} dv_\parallel = \int R_r dv_\parallel + \mathcal{G}_r S_r = \int \frac{1}{2}\mathcal{D}_1^+\mathcal{D}F_r^{(0)}dv_\parallel\,\partial_\xi\frac{|b_\perp^{(0)}|^2}{2B_0^2} - \int \frac{v_\perp}{2}\frac{\partial F_r^{(0)}}{\partial v_\perp}dv_\parallel\,\partial_\xi A$$
$$+\mathcal{G}_r\Big(\frac{v_\perp^2}{2} - \frac{q_r}{m_r}\mathcal{L}^{-1}\mathcal{M}\Big)\partial_\xi A. \tag{52}$$

It is at this step convenient to compute two quantities arising in the computation of the longitudinal and transverse pressure fluctuations performed in Section III. Defining

$$\mathcal{N} = 2\pi\sum_r m_r n_r \int_0^\infty d(\frac{v_\perp^2}{2})\frac{v_\perp^4}{4}\mathcal{G}_r, \tag{53}$$

one writes

$$\sum_r n_r m_r \int (v_\parallel - \lambda)^2 \partial_\xi \overline{f}_r^{(1)} d^3v = \sum_r n_r m_r \int (v_\parallel-\lambda)^2 R_r d^3v - \sum_r n_r m_r \int S_r F_r^{(0)} d^3v$$
$$= \frac{B_0^2}{4\pi}\partial_\xi A - \frac{B_0^2}{2\pi}\partial_\xi\frac{|b_\perp^{(0)}|^2}{2B_0^2} \tag{54}$$

and

$$\sum_r n_r m_r \int \frac{v_\perp^2}{2}\partial_\xi \overline{f}_r^{(1)} d^3v = 2\pi\sum_r n_r m_r \int d(\frac{v_\perp^2}{2})\frac{v_\perp^2}{2}\int \partial_\xi\overline{f}_r^{(1)} dv_\parallel$$
$$= \frac{B_0^2}{4\pi}\partial_\xi\frac{|b_\perp^{(0)}|^2}{2B_0^2} + (2p_\perp^{(0)} + \mathcal{N} - \mathcal{M}^2\mathcal{L}^{-1})\,\partial_\xi A. \tag{55}$$

D. The Kinetic DNLS equation

Proceeding like in Section IIB but at the next order, one now applies the operator $\sum_r m_r n_r \int d^3v\, \vec{v}_\perp \cdot$ on eq. (11) and evaluates the various terms in the resulting equation.

Using eq. (A.6), one gets

$$\sum_r m_r n_r \Omega_r \int \vec{v}_\perp \partial_\phi f_r^{(4)} d^3v = \frac{B_0}{4\pi}\left(\partial_\xi b_\perp^{(2)} - \nabla_\perp b_\parallel^{(1)}\right). \tag{56}$$

One also easily obtains

$$\sum_r m_r n_r \int \vec{v}_\perp \partial_\tau f_r^{(0)} d^3v = -\frac{\lambda\rho^{(0)}}{B_0}\partial_\tau b_\perp^{(0)} \tag{57}$$

and

$$\sum_r m_r n_r \int \vec{v}_\perp(\vec{v}_\perp\cdot\nabla_\perp) f_r^{(1)} d^3v =$$
$$\nabla_\perp \sum_r m_r n_r \int \frac{v_\perp^2}{2}\overline{f}_r^{(1)} d^3v - \frac{1}{4}\sum_r m_r n_r \int v_\perp^2 \begin{pmatrix} \sin 2\phi\,\partial_y - \cos 2\phi\,\partial_z \\ -\cos 2\phi\,\partial_y - \sin 2\phi\,\partial_z \end{pmatrix} \partial_\phi f_r^{(1)} d^3v \tag{58}$$

where the second term T_2 (including the sign) of the RHS is estimated by means of (B.3) and (B.4). Using eq. (30), one gets

$$T_2 = \frac{1}{8\pi}\begin{pmatrix} \partial_y(b_y^{(0)2} - b_z^{(0)2}) + 2\partial_z(b_y^{(0)} b_z^{(0)}) \\ -\partial_z(b_y^{(0)2} - b_z^{(0)2}) + 2\partial_y(b_y^{(0)} b_z^{(0)}) \end{pmatrix}. \tag{59}$$

Furthermore, as a consequence of the neutrality condition and the expension of the electric current (see Appendix A), one has

$$\sum_r m_r n_r \frac{q_r}{m_r}\int \vec{v}_\perp \left(e^{(4)} + \frac{1}{c} v \times b^{(4)}\right)\cdot\nabla_v F_r^{(0)} d^3v = 0 \tag{60}$$

and

$$\begin{aligned}
&\sum_{p+q=3}\sum_r m_r n_r \frac{q_r}{m_r}\int \vec{v}_\perp \left(e^{(p)} + \frac{1}{c} v\times b^{(p)}\right)\cdot\nabla_v f_r^{(q)} d^3v \\
&= -\sum_r q_r n_r \int \frac{1}{c}\left[(v_\parallel \widehat{x}\times b_\perp^{(0)}) f_r^{(3)} + (\vec{v}_\perp \times b_\parallel^{(1)}) f_r^{(2)}\right] d^3v \\
&= \frac{1}{4\pi}\left[b_\perp^{(0)}\times(\nabla_\perp\times b_\perp^{(0)}) - b_\parallel^{(1)}\partial_\xi b_\perp^{(0)}\right] \\
&= -\frac{1}{4\pi}\Big[(\partial_y b_z^{(0)} - \partial_z b_y^{(0)})\,\widehat{x}\times b_\perp^{(0)} + b_\parallel^{(1)}\partial_\xi b_\perp^{(0)}\Big].
\end{aligned} \tag{61}$$

This contribution is easily added to T_2. Using the divergenceless condition (21), the sum reduces to $-\frac{1}{4\pi}\partial_\xi(b_\parallel^{(1)} b_\perp^{(0)})$. Finally, using (B.5) and (B.6),

$$\begin{aligned}
&\sum_r m_r n_r \int \vec{v}_\perp (v_\parallel - \lambda)\partial_\xi f_r^{(2)} d^3v = \sum_r m_r n_r \int v_\perp (v_\parallel - \lambda)\partial_\xi (f_r^{(2)}\vec{t})\, d^3v \\
&= +\frac{\lambda}{B_0\Omega_i}(3p_{\parallel i}^{(0)} + \lambda^2\rho_i^{(0)} - 2p_{\perp i}^{(0)})\partial_{\xi\xi}(\widehat{x}\times b_\perp^{(0)}) + \frac{B_0}{4\pi}\partial_\xi b_\perp^{(2)} - \frac{1}{4\pi}\partial_\xi(b_\parallel^{(1)} b_\perp^{(0)}) - \frac{\lambda\rho^{(0)}}{B_0}\partial_\tau b_\perp^{(0)} \\
&\quad + \frac{1}{8\pi B_0}\partial_\xi(|b_\perp^{(0)}|^2 b_\perp^{(0)}) + \frac{1}{B_0}\partial_\xi\Big(\sum_r m_r n_r \int \Big[(v_\parallel - \lambda)^2 - \frac{v_\perp^2}{2}\Big]\overline{f}_r^{(1)} d^3v\; b_\perp^{(0)}\Big),
\end{aligned} \tag{62}$$

where only the contribution of the ions was retained in the dispersive term. Summing up the various contributions computed above, one notes that the terms involving $b_\perp^{(2)}$ cancel out. Writing $\overline{f}_r^{(1)} = \langle \overline{f}_r^{(1)}\rangle_\xi + \widetilde{\overline{f}}_r^{(1)}$ where the brackets

$\langle \cdot \rangle_\xi = \lim_{L\to\infty} \frac{1}{2L}\int_{-L}^{+L} \cdot \, d\xi$ indicate averaging along the direction of the ambient field and the tilded quantities are the fluctuations about this mean value, one sees that $\langle \overline{f}_r^{(1)} \rangle_\xi$ contributes to eq. (62).

One gets the dynamical equation

$$\partial_\tau b_\perp^{(0)} + \delta \partial_{\xi\xi}(\widehat{x} \times b_\perp^{(0)}) - B_0 \nabla_\perp \Big(\frac{\widetilde{P}}{2\lambda\rho^{(0)}} \Big) + \frac{\partial}{\partial \xi}\Big[\Big(\frac{\widetilde{P}}{2\lambda\rho^{(0)}} + \langle U \rangle_\xi \Big) b_\perp^{(0)} \Big] = 0, \tag{63}$$

with a dispersion coefficient δ given by

$$\delta = \frac{1}{2\Omega_i}\Big(\lambda^2 + 3\frac{p_{\parallel i}^{(0)}}{\rho^{(0)}} - 2\frac{p_{\perp i}^{(0)}}{\rho^{(0)}} \Big) = \frac{1}{2\Omega_i \rho^{(0)}}\Big(\frac{B_0^2}{4\pi} + 2p_{\parallel i}^{(0)} - p_{\perp i}^{(0)} - p_{\parallel e}^{(0)} - p_{\perp e}^{(0)} \Big). \tag{64}$$

Here the subscripts e and i refer to electrons and ions (assumed of a unique species) respectively. The ion density at equilibrium has furthermore been replaced by the total plama density. One also has defined

$$\widetilde{P} = \Big(\frac{B_0^2}{4\pi} + 2p_\perp^{(0)} + \mathcal{N} - \mathcal{M}^2 \mathcal{L}^{-1} \Big) \widetilde{A} \tag{65}$$

where the magnetic fluctuations $\widetilde{b}_\parallel^{(1)}$ along the ambient field is prescribed in terms of the transverse component $b_\perp^{(0)}$ by the divergenceless condition (21). When dealing with Alfvén pulses whose extension is comparable to the longitudinal scale ϵ^{-2} retained in the definition of the parallel stretched coordinate, mean fields are zero and the system is closed. The resulting system was derived by Rogister[3] using a Fourier space formalism. The presence of Landau damping has important physical consequences. In particular, the criteria for modulational instability in the direction of propagation are strongly modified by this effect effect[6,19] and new dissipative structures were also reported.[20] These authors also notice the suppression of the resonance predicted by the fluid theory for $\beta = 1$, that breaks the DNLS scalings.[21] A quasi-resonance survives only in the special case when the temperature of the electrons strongly exceeds that of the ions.[22] Note that the action of the Landau damping on the Alfvén waves is mediated by the coupling with ion-acoustic waves that are directly affected.

When dealing with Alfvén wave trains that include a large number of pump wavelengths of order ϵ^{-2} the mean fields are no longer negligible. It is given by

$$\langle U \rangle_\xi = -\frac{B_0^2}{8\pi\lambda\rho^{(0)}}\Big(\langle \frac{|b_\perp^{(0)}|^2}{2B_0^2} \rangle_\xi - 2 \langle \frac{b_\parallel^{(1)}}{B_0} \rangle_\xi \Big) - \frac{1}{2\lambda\rho^{(0)}} \sum_r m_r n_r \int \Big[(v_\parallel - \lambda)^2 - \frac{v_\perp^2}{2} \Big] \langle \overline{f}_r^{(1)} \rangle_\xi d^3 v \tag{66}$$

that can be viewed as a convective velocity which locally corrects the Alfvén wave speed. Its computation requires the estimate of $\langle \overline{f}_r^{(1)} \rangle_\xi$ by pushing to higher order the expansion of the Vlasov-Maxwell system.

Furthermore, eq. (63) requires the solvability condition

$$\frac{B_0^2}{4\pi} \langle \frac{b_\parallel^{(1)}}{B_0} \rangle_\xi + \sum_r m_r n_r \int \frac{v_\perp^2}{2} \langle \overline{f}_r^{(1)} \rangle_\xi d^3 v = \Gamma(\tau) \tag{67}$$

where $\Gamma(\tau)$ denotes a function of τ only that will be determined in the following. Indeed, the quantity $\nabla_\perp \Gamma$ arises as a source term in the equation for the transverse magnetic field $b_\perp^{(0)}$. Since, from eq. (21), $\nabla_\perp \cdot \Big\langle b_\perp^{[0]} \Big\rangle_\xi = 0$, one obtains $\Delta_\perp \Gamma = 0$. For solutions decaying at large transverse distance or obeying periodic boundary conditions, no mean field $\langle b_\perp^{(0)} \rangle_\xi$ is thus driven.

III. RELATION WITH HYDRODYNAMIC QUANTITIES

For a physical interpretation of the various contributions to the long-wave equation, it is useful to compute a few hydrodynamic quantities and in particular the pressure perturbations in the directions transverse and parallel to the *local* magnetic field. Note that the pressure disturbances arise at an order of perturbation where the local distortion of the magnetic field line cannot be neglected. The fluctuating parts of the heat fluxes that play an important role in Paper II are also computed in this section.

A. Hydrodynamic velocities

The velocity component $V_\perp = v - \frac{(v\cdot b)}{|b|^2} b$ transverse to the local magnetic field $b = \Big(B_0 + \epsilon^2 b_\parallel^{(1)}\ ,\ \epsilon b_\perp^{(0)}\Big) + O(\epsilon^3)$ reads

$$V_\perp = \Big(- \frac{\epsilon}{B_0}(\vec{v}_\perp \cdot b_\perp^{(0)}) + \epsilon^2 v_\parallel \frac{|b_\perp^{(0)}|^2}{B_0^2}\ ,\ \vec{v}_\perp - \epsilon \frac{v_\parallel}{B_0} b_\perp^{(0)} - \epsilon^2 \frac{(\vec{v}_\perp \cdot b_\perp^{(0)})}{B_0^2} b_\perp^{(0)} \Big) + O(\epsilon^3). \tag{68}$$

The corresponding hydrodynamic velocity $U_\perp = \frac{\sum_r m_r n_r \int V_\perp f_r d^3v}{\sum_r m_r n_r \int f_r d^3 v}$ is thus

$$U_\perp = \Big(\lambda \frac{|b_\perp^{(0)}|^2}{B_0^2}\epsilon^2\ ,\ -\frac{\lambda\epsilon}{B_0} b_\perp^{(0)} \Big) + O(\epsilon^3). \tag{69}$$

The leading order component in the RHS of eq. (69) is the characteristic signature of an Alfvén wave and also identifies with the electric drift velocity $c(e_\perp^{(0)} \times B_0 \widehat{x})/B_0^2$.

The velocity component along the local magnetic field $V_\parallel = \frac{v\cdot b}{|b|}$ is given by

$$V_\parallel = v_\parallel + \frac{\epsilon}{B_0}(\vec{v}\cdot b_\perp^{(0)}) - \frac{\epsilon^2 v_\parallel}{2B_0^2}|b_\perp^{(0)}|^2 + O(\epsilon^3), \tag{70}$$

and the corresponding component of the hydrodynamic velocity $U_\parallel = \frac{\sum_r m_r n_r \int V_\parallel f_r d^3 v}{\sum_r m_r n_r \int f_r d^3 v}$ is

$$\begin{aligned} U_\parallel &= \frac{\epsilon^2}{\rho^{(0)}} \sum_r m_r n_r \int \Big[v_\parallel \overline{f}_r^{(1)} + \frac{1}{B_0}(\vec{v}_\perp \cdot b_\perp^{(0)}) f_r^{(0)} \Big] d^3 v + O(\epsilon^3) \\ &= \epsilon^2 \Big(\frac{1}{\rho^{(0)}} \sum_r m_r n_r \int v_\parallel \overline{f}_r^{(1)} d^3 v - \lambda \frac{|b_\perp^{(0)}|^2}{B_0^2} \Big) + O(\epsilon^3). \end{aligned} \tag{71}$$

Furthermore, the fluctuating hydrodynamic velocity $u_\parallel = \epsilon^2 u_\parallel^{(1)} + O(\epsilon^3)$ in the direction of propagation (or equivalently along the ambient field) is given by

$$\widetilde{u}_\parallel^{(1)} = \frac{1}{\rho^{(0)}} \sum_r m_r n_r \int v_\parallel \widetilde{\overline{f}}_r^{(1)} d^3 v = \frac{1}{\rho^{(0)}} \sum_r m_r n_r \int (v_\parallel - \lambda) \widetilde{\overline{f}}_r^{(1)} d^3 v + \lambda \frac{\widetilde{\rho}^{(1)}}{\rho^{(0)}}, \tag{72}$$

where from eq. (38),

$$\sum_r m_r n_r \int (\lambda - v_\parallel) \widetilde{\overline{f}}_r^{(1)} d^3 v = \lambda \rho^{(0)} \frac{\widetilde{b}_\parallel^{(1)}}{B_0}. \tag{73}$$

One thus gets the relation also obtained in the Hall-MHD context[9]

$$\frac{\widetilde{u}_\parallel^{(1)}}{\lambda} - \frac{\widetilde{\rho}^{(1)}}{\rho^{(0)}} + \frac{\widetilde{b}_\parallel^{(1)}}{B_0} = 0. \tag{74}$$

B. Transverse pressure

One can expand the transverse pressure $p_\perp = \sum_r m_r n_r \int \frac{1}{2}(V_\perp - U_\perp)^2 f_r d^3 v$ in the form $p_\perp = p_\perp^{(0)} + \epsilon^2 p_\perp^{(1)} + O(\epsilon^3)$ with $p_\perp^{(1)} = \langle p_\perp^{(1)} \rangle_\xi + \widetilde{p}_\perp^{(1)}$. One easily gets

$$\begin{aligned} p_\perp^{(1)} &= \sum_r m_r n_r \int \Big\{ \frac{v_\perp^2}{2} \overline{f}_r^{(1)} + (\lambda - v_\parallel)\frac{(\vec{v}_\perp \cdot b_\perp^{(0)})}{B_0} f_r^{(0)} + \frac{1}{2}\Big[(\lambda - v_\parallel)^2 \frac{|b_\perp^{(0)}|^2}{B_0^2} - \frac{(\vec{v}_\perp \cdot b_\perp^{(0)})^2}{B_0^2} \Big] F_r^{(0)} \Big\} d^3 v \\ &= \sum_r m_r n_r \int \frac{v_\perp^2}{2} \overline{f}_r^{(1)} d^3 v - \Big(\frac{B_0^2}{4\pi} \Big) \frac{|b_\perp^{(0)}|^2}{2B_0^2}. \end{aligned} \tag{75}$$

It follows that

$$\widetilde{p}_\perp^{(1)} = (2p_\perp^{(0)} + \mathcal{N} - \mathcal{M}^2\mathcal{L}^{-1})\widetilde{A} \tag{76}$$

and

$$\langle p_\perp^{(1)}\rangle_\xi = \sum_r m_r n_r \int \frac{v_\perp^2}{2}\langle \overline{f}_r^{(1)}\rangle_\xi d^3v - \left(\frac{B_0^2}{4\pi}\right)\frac{\langle |b_\perp^{(0)}|^2\rangle_\xi}{2B_0^2}. \tag{77}$$

Note that $\widetilde{p}_\perp^{(1)}$ contributes to $\widetilde{P}$ that enters eq. (63) for the transverse magnetic field.

C. Density and longitudinal pressure

The longitudinal pressure is $p_\| = \sum_r m_r n_r \int (V_\| - U_\|)^2 f_r d^3v = p_\|^{(0)} + \epsilon^2 p_\|^{(1)} + O(\epsilon^3)$ with

$$\begin{aligned} p_\|^{(1)} &= \sum_r m_r n_r \int \Big(\frac{1}{B_0^2}\Big[(\vec{v}_\perp \cdot b_\perp^{(0)})^2 - v_\|^2 |b_\perp^{(0)}|^2\Big] F_r^{(0)} + 2\frac{v_\|}{B_0}(\vec{v}_\perp \cdot b_\perp^{(0)}) f_r^{(0)} + v_\|^2 \overline{f}_r^{(1)}\Big)\, d^3v \\ &= (p_\|^{(0)} - p_\perp^{(0)})\frac{|b_\perp^{(0)}|^2}{B_0^2} + \sum_r m_r n_r \int v_\|^2\, \overline{f}_r^{(1)}\, d^3v. \end{aligned} \tag{78}$$

Note that $U_\|$ does not contribute to $p_\|^{(1)}$.

Writing $p_\|^{(1)} = \langle p_\|^{(1)}\rangle_\xi + \widetilde{p_\|^{(1)}}$, one has

$$\langle p_\|^{(1)}\rangle_\xi = (p_\|^{(0)} - p_\perp^{(0)})\frac{\langle |b_\perp^{(0)}|^2\rangle_\xi}{B_0^2} + \sum_r m_r n_r \int v_\|^2\, \langle \overline{f}_r^{(1)}\rangle_\xi\, d^3v. \tag{79}$$

Defining the operators

$$\mathcal{O} = 2\pi \sum_r m_r n_r \int_0^\infty \frac{v_\perp^2}{2}\mathcal{G}_r d(\frac{v_\perp^2}{2}) \quad , \quad \mathcal{P} = 2\pi \sum_r q_r n_r \int_0^\infty \mathcal{G}_r d(\frac{v_\perp^2}{2}), \tag{80}$$

one also expresses the leading order contribution to the fluctuating density and parallel pression perturbations in the form

$$\widetilde{\rho}^{(1)} = \sum_r m_r n_r \int \widetilde{\overline{f}}_r^{(1)}\, d^3v = (\rho^{(0)} + \mathcal{O} - \mathcal{P}\mathcal{L}^{-1}\mathcal{M})\widetilde{A} \tag{81}$$

$$\begin{aligned} \widetilde{p}_\|^{(1)} &= [p_\|^{(0)} - p_\perp^{(0)} + \lambda^2(\mathcal{O} - \mathcal{P}\mathcal{L}^{-1}\mathcal{M})]\widetilde{A} \\ &= (p_\|^{(0)} - p_\perp^{(0)})\widetilde{A} + \lambda^2(\widetilde{\rho}^{(1)} - \rho^{(0)}\widetilde{A}). \end{aligned} \tag{82}$$

D. Heat fluxes

The transverse heat flux is defined as $q_\perp = \sum_r m_r n_r \int \frac{1}{2}(V_\perp - U_\perp)^2 (V_\| - U_\|) f_r d^3v$. To leading order, $q_\perp = \epsilon^2 q_\perp^{(1)} + O(\epsilon^3)$. A straightforward expansion leads to

$$q_\perp^{(1)} = \frac{1}{2}\Big[\sum_r m_r n_r \Big(\int v_\perp^2 v_\| \overline{f}_r^{(1)} d^3v - 2\frac{p_\perp^{(0)}}{\rho^{(0)}}\int v_\| \overline{f}_r^{(1)} d^3v\Big) + 2\lambda(p_\|^{(0)} - p_\perp^{(0)})\frac{|b_\perp^{(0)}|^2}{B_0^2}\Big]. \tag{83}$$

In order to compute the fluctuating part of the transverse heat flux, one uses eqs. (38)-(42) and finally gets

$$\widetilde{q}_\perp^{(1)} = \lambda p_\perp^{(0)}\left(\frac{\widetilde{p}_\perp^{(1)}}{p_\perp^{(0)}} - \frac{\widetilde{\rho}^{(1)}}{\rho^{(0)}} - \widetilde{A}\right). \tag{84}$$

The parallel heat flux is $q_\parallel = \sum_r m_r n_r \int (V_\parallel - U_\parallel)^3 f_r d^3v$. Again to leading order, $q_\parallel = \epsilon^2 q_\parallel^{(1)} + O(\epsilon^3)$. Expanding as above, one gets

$$q_\parallel^{(1)} = \sum_r m_r n_r \Big[\int v_\parallel^3 \overline{f}_r^{(1)} d^3v + \int 3 v_\parallel^2 \frac{v_\perp \cdot b_\perp^{(0)}}{B_0} f_r^{(0)} d^3v \Big]$$
$$+6\lambda p_\parallel^{(0)} \frac{|b_\perp^{(0)}|^2}{2B_0^2} - 3p_\parallel^{(0)} \frac{1}{\rho^{(0)}} \sum_r m_r n_r \int v_\parallel \overline{f}_r^{(1)} d^3v. \tag{85}$$

For the fluctuating part, one has

$$\widetilde{q}_\parallel^{(1)} = \lambda \Big[3p_\parallel^{(0)} \frac{\widetilde{|b_\perp^{(0)}|^2}}{2B_0^2} - (p_\perp^{(0)} + \lambda^2 \rho^{(0)}) \widetilde{A} + \lambda^2 \widetilde{\rho}^{(1)} - 3p_\perp^{(0)} \frac{\widetilde{u}_\parallel^{(1)}}{\lambda} \Big] \tag{86}$$

or

$$\widetilde{q}_\parallel^{(1)} = \lambda p_\parallel^{(0)} \left(\frac{\widetilde{p}_\parallel^{(1)}}{p_\parallel^{(0)}} - 3 \frac{\widetilde{\rho}^{(1)}}{\rho^{(0)}} + 2\widetilde{A} \right). \tag{87}$$

IV. EVOLUTION OF THE MEAN FIELDS

A. Mean field contributions in the KDNLS equation

The term $-\frac{1}{2\lambda\rho^{(0)}} \sum_r m_r n_r \int [(v_\parallel - \lambda)^2 - \frac{v_\perp^2}{2}] \langle \overline{f}_r^{(1)} \rangle_\xi d^3v$ in eq. (66) is expressed in a convenient form when introducing the mean velocity in the direction of propagation (or of the ambient field)

$$\langle u_\parallel^{(1)} \rangle_\xi = \sum_r m_r n_r \frac{1}{\rho^{(0)}} \int v_\parallel \ \langle \overline{f}_r^{(1)} \rangle_\xi d^3v, \tag{88}$$

and also the mean perturbation of the plasma density

$$\langle \rho^{(1)} \rangle_\xi = \sum_r m_r n_r \int \langle \overline{f}_r^{(1)} \rangle_\xi d^3v, \tag{89}$$

both arising as $O(\epsilon^2)$ contributions. One easily shows, using eqs. (79) and (77) that

$$-\frac{1}{2\lambda\rho^{(0)}} \sum_r m_r n_r \int \left[(v_\parallel - \lambda)^2 - \frac{v_\perp^2}{2} \right] \langle \overline{f}_r^{(1)} \rangle_\xi d^3v =$$
$$\frac{1}{2\lambda\rho^{(0)}} \Big(\langle p_\perp^{(1)} \rangle_\xi - \langle p_\parallel^{(1)} \rangle_\xi \Big) + \langle u_\parallel^{(1)} \rangle - \frac{\lambda}{2} \frac{\langle \rho^{(1)} \rangle_\xi}{\rho^{(0)}} + \frac{1}{\lambda\rho^{(0)}} \Big(p_\parallel^{(0)} - p_\perp^{(0)} + \frac{B_0^2}{8\pi} \Big) \frac{\langle |b_\perp^{(0)}|^2 \rangle_\xi}{2B_0^2}. \tag{90}$$

On the other hand, using eq. (77), the solvability condition (67) is expressed in terms of the mean pressure fluctuations transverse to the local magnetic field in the form

$$\langle p_\perp^{(1)} \rangle_\xi + \frac{B_0^2}{4\pi} \langle A \rangle_\xi = \Gamma(\tau) \tag{91}$$

where, as already mentioned, $\Gamma(\tau)$ is a function of time only that is to be determined. Equation (91) relates the mean longitudinal perturbation $\left\langle b_\parallel^{(1)} \right\rangle_\xi$ to $\frac{\langle |b_\perp^{(0)}|^2 \rangle_\xi}{2B_0^2}$.

B. Determination of the mean density

Starting from the definition of the mean density $\langle\rho^{(1)}\rangle_\xi$, one uses eq. (12) and gets

$$\partial_\tau\langle\rho^{(1)}\rangle_\xi = -\sum_r m_r n_r \int v_\perp \left(\partial_y\langle\langle\cos\phi f_r^{(2)}\rangle\rangle_\xi + \partial_z\langle\langle\sin\phi f_r^{(2)}\rangle\rangle_\xi\right) d^3v. \tag{92}$$

If follows from eqs. (B.5) and (B.6) that

$$\begin{aligned}\partial_y\langle\langle\cos\phi f_r^{(2)}\rangle\rangle_\xi + \partial_z\langle\langle\sin\phi f_r^{(2)}\rangle\rangle_\xi = &-\frac{1}{2B_0^2}\nabla_\perp\cdot\langle b_\parallel^{(1)} b_\perp^{(0)}\rangle_\xi \mathcal{D}F_r^{(0)} + \frac{1}{2B_0}\partial_\tau\langle b_\parallel^{(1)}\rangle_\xi \frac{\partial F_r^{(0)}}{\partial v_\perp}\\ &-\frac{1}{16B_0^3}\nabla_\perp\cdot\langle|b_\perp^{(0)}|^2 b_\perp^{(0)}\rangle_\xi\ \mathcal{D}_2^+\mathcal{D}_1^-\mathcal{D}F_r^{(0)} + \frac{1}{2B_0}\nabla_\perp\cdot\langle b_\perp^{(0)}\mathcal{D}\overline{f}_r^{(1)}\rangle_\xi.\end{aligned} \tag{93}$$

Note that $\overline{f}_r^{(1)}$ contributes only by its fluctuations $\widetilde{\overline{f}}_r^{(1)}$. Furthermore, due to the presence of the operator $\mathcal{D}$ acting on $\overline{f}_r^{(1)}$ within the integral on the velocities, no singular contributions arises. It is thus convenient to use eq. (38) and one finally gets the simple relation

$$\partial_\tau\frac{\langle\rho^{(1)}\rangle_\xi}{\rho^{(0)}} = \partial_\tau\frac{\langle b_\parallel^{(1)}\rangle_\xi}{B_0}. \tag{94}$$

C. Evolution of the mean longitudinal velocity

Using again eq. (12), one expresses the time derivative of the mean hydrodynamic velocity in the direction of propagation in the form

$$\begin{aligned}\rho^{(0)}\partial_\tau\langle u_\parallel^{(1)}\rangle_\xi = &-\sum_r m_r n_r \int v_\parallel v_\perp \left(\partial_y\langle\langle\cos\phi f_r^{(2)}\rangle\rangle_\xi + \partial_z\langle\langle\sin\phi f_r^{(2)}\rangle\rangle_\xi\right) d^3v\\ &-\sum_r q_r n_r \int v_\parallel \Big\langle \frac{\vec{v}_\perp\times b_\perp^{(4)}}{c}\frac{\partial f_r^{(0)}}{\partial v_\parallel} + \frac{\vec{v}_\perp\times b_\perp^{(2)}}{c}\frac{\partial f_r^{(2)}}{\partial v_\parallel} + \frac{\vec{v}_\perp\times b_\perp^{(0)}}{c}\frac{\partial f_r^{(0)}}{\partial v_\parallel}\Big\rangle_\xi d^3v.\end{aligned} \tag{95}$$

In the second integral of the RHS, the term involving $b_\perp^{(4)}$ does not give any contributions. The two other terms are estimated using the expansion of the electric current presented in Appendix B. The second term gives $-\frac{1}{4\pi}b_\perp^{(2)}\times(\widehat{x}\times\partial_\xi b_\perp^{(0)}) = -\frac{1}{4\pi}b_\perp^{(2)}\cdot\partial_\xi b_\perp^{(0)}$. Similarly, the third term is $-\frac{1}{4\pi}\langle b_\perp^{(0)}\times(\widehat{x}\times\partial_\xi b_\perp^{(2)}) + \nabla_\perp\times b_\parallel^{(1)}\rangle_\xi = -\frac{1}{4\pi}\langle b_\perp^{(0)}\cdot\partial_\xi b_\perp^{(2)} - (b_\perp^{(0)}\cdot\nabla_\perp)b_\parallel^{(1)}\rangle_\xi$ where, from eq. (21), $\langle b_\perp^{(0)}\cdot\nabla_\perp)b_\parallel^{(1)}\rangle_\xi = \nabla_\perp\cdot\langle b_\parallel^{(1)} b_\perp^{(0)}\rangle_\xi$. It follows that

$$\rho^{(0)}\partial_\tau\langle u_\parallel^{(1)}\rangle_\xi = -\sum_r m_r n_r \int v_\parallel v_\perp \left(\partial_y\langle\langle\cos\phi f_r^{(2)}\rangle\rangle_\xi + \partial_z\langle\langle\sin\phi f_r^{(2)}\rangle\rangle_\xi\right) d^3v + \frac{1}{4\pi}\nabla_\perp\cdot\langle b_\parallel^{(1)} b_\perp^{(0)}\rangle_\xi \tag{96}$$

where the first term of the RHS is estimated by means of eq. (93). As a consequence,

$$\begin{aligned}&-\sum_r m_r n_r \int v_\parallel v_\perp \left(\partial_y\langle\langle\cos\phi f_r^{(2)}\rangle\rangle_\xi + \partial_z\langle\langle\sin\phi f_r^{(2)}\rangle\rangle_\xi\right) d^3v\\ &= \frac{1}{B_0^2}(p_\parallel^{(0)} - p_\perp^{(0)})\nabla_\perp\cdot\langle b_\parallel^{(1)} b_\perp^{(0)}\rangle_\xi - \frac{1}{8\pi B_0}\nabla_\perp\cdot\langle|b_\perp^{(0)}|^2 b_\perp^{(0)}\rangle_\xi\\ &+\frac{1}{B_0}\nabla_\perp\cdot\Big\langle b_\perp^{(0)}\sum_r m_r n_r\int\Big[\lambda(\lambda - v_\parallel) - (\lambda - v_\parallel)^2 + \frac{v_\perp^2}{2}\Big]\overline{f}_r^{(1)} d^3v\rangle_\xi.\end{aligned} \tag{97}$$

Taking into account that $\langle b_\perp^{(0)}\rangle_\xi = 0$ and using (73), (54) and (55), one finally gets

$$\partial_\tau\langle u_\parallel^{(1)}\rangle_\xi = \frac{1}{\rho^{(0)}B_0}\nabla_\perp\cdot\langle\widetilde{P} b_\perp^{(0)}\rangle_\xi \tag{98}$$

with $\widetilde{P}$ defined in (65).

D. Evolution of the mean pressures

In order to close the system, it is necessary to determine the evolution of $\sum_r m_r n_r \int \frac{v_\perp^2}{2}\langle \overline{f}_r^{(1)}\rangle_\xi \, d^3v$ and of

$$\sum_r m_r n_r \int (v_\parallel-\lambda)^2 \langle \overline{f}_r^{(1)}\rangle_\xi \, d^3v = \sum_r m_r n_r \int v_\parallel^2 \langle \overline{f}_r^{(1)}\rangle_\xi \, d^3v - 2\lambda\rho^{(0)}\langle u_\parallel^{(1)}\rangle_\xi + \lambda^2\langle\rho^{(1)}\rangle_\xi. \tag{99}$$

Using (12), one writes

$$\begin{aligned}\partial_\tau \langle \overline{f}_r^{(1)}\rangle_\xi &= -v_\perp \Big(\partial_y\langle\langle \cos\phi f_r^{(2)}\rangle\rangle_\xi + \partial_z\langle\langle \sin\phi f_r^{(2)}\rangle\rangle_\xi\Big)\\ &-\frac{q_r}{m_r}\langle\langle\Big(e^{(5)}+\frac{v\times b^{(5)}}{c}\Big)\cdot\nabla_v F_r^{(0)}\rangle\rangle_\xi - \frac{q_r}{m_r}\sum_{p+q=4}\langle\langle\Big(e^{(p)}+\frac{v\times b^{(p)}}{c}\Big)\cdot\nabla_v f_r^{(q)}\rangle\rangle_\xi,\end{aligned} \tag{100}$$

that is substituted in $\sum_r m_r n_r \int \frac{v_\perp^2}{2}\partial_\tau\langle \overline{f}_r^{(1)}\rangle_\xi \, d^3v$ and $\sum_r m_r n_r \int v_\parallel^2 \partial_\tau \langle \overline{f}_r^{(1)}\rangle_\xi \, d^3v$. The first term is given by eq. (93) and requires to estimate the coefficients $\sum_r m_r n_r \int v_\perp^3 \mathcal{D}F_r^{(0)} d^3v = -8\lambda p_\perp^{(0)}$, $\sum_r m_r n_r \int v_\perp^3 \frac{\partial F_r^{(0)}}{\partial v_\perp} d^3v = -8p_\perp^{(0)}$, together with $\sum_r m_r n_r \int v_\perp^3 \mathcal{D}_2^+\mathcal{D}_1^-\mathcal{D}F_r^{(0)} d^3v = -16\lambda\Big(\frac{B_0^2}{4\pi}+2(p_\parallel^{(0)}-p_\perp^{(0)})\Big)$, $\sum_r m_r n_r \int v_\parallel^2 v_\perp \mathcal{D}F_r^{(0)} d^3v = -2\lambda p_\parallel^{(0)}$, $\sum_r m_r n_r \int v_\parallel^2 v_\perp \frac{\partial F_r^{(0)}}{\partial v_\perp} d^3v = -2p_\parallel^{(0)}$ and $\sum_r m_r n_r \int v_\parallel^2 v_\perp \mathcal{D}_2^+\mathcal{D}_1^-\mathcal{D}F_r^{(0)} d^3v = 32\lambda\left(p_\parallel^{(0)}-p_\perp^{(0)}\right)$. Furthermore, from (39), (40) and (41), one has

$$\sum_r m_r n_r \int v_\perp^3 \mathcal{D}\overline{f}_r^{(1)} d^3v = -16\lambda p_\perp^{(0)} A - 8\lambda\Big(\frac{B_0^2}{4\pi}+2p_\parallel^{(0)}-3p_\perp^{(0)}\Big)\frac{|b_\perp^{(0)}|^2}{2B_0^2} \tag{101}$$

and

$$\begin{aligned}\sum_r m_r n_r \int v_\parallel^2 v_\perp \mathcal{D}\widetilde{\overline{f}}_r^{(1)} d^3v &= -2\lambda\left(p_\parallel^{(0)}-4p_\perp^{(0)}\right)A\\ &+2\lambda\Big(2\frac{B_0^2}{4\pi}+9p_\parallel^{(0)}-8p_\perp^{(0)}\Big)\left(\frac{|b_\perp^{(0)}|^2}{2B_0^2}\right) - 2\lambda\sum_r m_r n_r \int v_\perp^2 \widetilde{\overline{f}}_r^{(1)} d^3v\end{aligned} \tag{102}$$

where the last term of the RHS is given by eq. (55).

When considering the contribution of the electric field, one notes that $e^{(1)}=0$ and also that $e^{(3)}$ and $e^{(5)}$ (which are parallel to the propagation) do not contribute to $\int v_\perp^2 \partial_\tau \langle \overline{f}_r^{(1)}\rangle_\xi \, d^3v$. nor to $\sum_r m_r n_r \int v_\parallel^2 \partial_\tau \langle \overline{f}_r^{(1)}\rangle_\xi \, d^3v$ as easily seen by integration by part and use of the fact that $f_r^{(1)}$ and $F_r^{(0)}$ do not enter the electric current (see Appendix A). Similarly, $e^{(4)}$ (that is transverse) does not contribute to $\sum_r m_r n_r \int v_\parallel^2 \partial_\tau \langle \overline{f}_r^{(1)}\rangle_\xi \, d^3v$ nor to $\int v_\perp^2 \partial_\tau \langle \overline{f}_r^{(1)}\rangle_\xi \, d^3v$. As a consequence, the electric field only enters $\sum_r m_r n_r \int v_\perp^2 \partial_\tau \langle \overline{f}_r^{(1)}\rangle_\xi \, d^3v$ through $e_\perp^{(0)}$ and $e_\perp^{(2)}$. One gets

$$\begin{aligned}&\Big\langle \sum_r q_r n_r \int v_\perp^2 (e_\perp^{(2)}\cdot\nabla_{v_\perp} f_r^{(2)} + (e_\perp^{(0)}\cdot\nabla_{v_\perp} f_r^{(4)}) d^3v\Big\rangle_\xi\\ &= -\frac{c}{2\pi}\Big\langle e_\perp^{(2)}\cdot\Big(\widehat{x}\times\partial_\xi b_\perp^{(0)}\Big) + e_\perp^{(0)}\cdot\Big(\nabla_\perp\times b_\parallel^{(1)} + \widehat{x}\times\partial_\xi b_\perp^{(2)}\Big)\Big\rangle_\xi\\ &= \frac{1}{4\pi}\partial_\tau\Big\langle |b_\perp^{(0)}|^2\Big\rangle_\xi - \frac{\lambda}{2\pi}\nabla_\perp\cdot\Big\langle b_\parallel^{(1)} b_\perp^{(0)}\Big\rangle_\xi.\end{aligned} \tag{103}$$

Furthermore, it is convenient to write

$$(v\times b^{(p)})\cdot\nabla_v f_r^{(q)} = (\vec{v}_\perp\times b_\parallel^{(p)})\cdot\nabla_{v_\perp} f_r^{(q)} + v_\parallel(\widehat{x}\times b_\perp^{(p)})\cdot\nabla_{v_\perp} f_r^{(q)} + (\vec{v}_\perp\times b_\perp^{(p)})\partial_{v_\parallel} f_r^{(q)}. \tag{104}$$

The first term of the RHS rewrites $-b_\parallel^{(p)}\partial_\phi f^{(q)}$ and thus does not contribute to the integrals under consideration. The second one contributes to $\int v_\perp^2 \partial_\tau \langle \overline{f}_r^{(1)}\rangle_\xi \, d^3v$ and the third one to $\int v_\parallel^2 \partial_\tau \langle \overline{f}_r^{(1)}\rangle_\xi \, d^3v$. One thus rewrites

$$\int v_\perp^2 \frac{(v\times b^{(p)})_\perp}{c}\cdot \nabla_{v_\perp} f_r^{(q)} d^3v = -\frac{2}{c} b_\perp^{(p)} \cdot \int v_\parallel v_\perp \langle \vec{n}\frac{\partial f_r^{(q)}}{\partial \phi}\rangle \, d^3v \tag{105}$$

$$\int v_\parallel^2 \frac{(v\times b^{(p)})_\perp}{c}\cdot \nabla_{v_\perp} f_r^{(q)} d^3v = \frac{2}{c} b_\perp^{(p)} \cdot \int v_\parallel v_\perp \langle \vec{n}\frac{\partial f_r^{(q)}}{\partial \phi}\rangle \, d^3v \tag{106}$$

where $\langle \vec{n}\frac{\partial f_r^{(q)}}{\partial \phi}\rangle$ with $q = 0$, 2 and 4 are computed in Appendix B.

One easily gets from eqs. (B.1) and (B.2) that

$$\Big\langle b_\perp^{(4)} \cdot \langle \vec{n}\frac{\partial f_r^{(0)}}{\partial \phi}\rangle\Big\rangle_\xi = \frac{1}{2B_0}\langle b_\perp^{(4)} \times b_\perp^{(0)}\rangle_\xi \mathcal{D} F_r^{(0)}. \tag{107}$$

Similarly, from (B.5) and (B.6), one has

$$\begin{aligned}\Big\langle b_\perp^{(2)} \cdot \langle \vec{n}\frac{\partial f_r^{(2)}}{\partial \phi}\rangle\Big\rangle_\xi &= -\frac{1}{2\Omega_r B_0}\langle b_\perp^{(2)}\cdot \partial_\xi b_\perp^{(0)}\rangle_\xi (\lambda - v_\parallel)\mathcal{D}F_r^{(0)} \\ &-\frac{1}{2B_0^2}\langle b_\parallel^{(1)} b_\perp^{(2)} \times b_\perp^{(0)}\rangle_\xi \mathcal{D}F_r^{(0)} - \frac{1}{2B_0}\langle b_\perp^{(2)} \times \partial_\xi^{-1}\partial_\tau b_\perp^{(0)}\rangle_\xi \frac{\partial F_r^{(0)}}{\partial v_\perp} \\ &-\frac{1}{16B_0^3}\langle |b_\perp^{(0)}|^2 b_\perp^{(2)} \times b_\perp^{(0)}\rangle_\xi \mathcal{D}_2^+\mathcal{D}_1^-\mathcal{D}F_r^{(0)} + \frac{1}{2B_0}\langle b_\perp^{(2)} \times b_\perp^{(0)} \mathcal{D}\overline{f}_r^{(1)}\rangle_\xi .\end{aligned} \tag{108}$$

Furthermore, using again Appendix B,

$$\begin{aligned}\Big\langle b_\perp^{(0)} \cdot \langle \vec{n}\frac{\partial f_r^{(4)}}{\partial \phi}\rangle\Big\rangle_\xi &= \frac{1}{\Omega_r}\Big\{(v_\parallel - \lambda)\Big[\frac{1}{2B_0}\Big\langle b_\perp^{(0)}\cdot\partial_\xi b_\perp^{(2)}\Big\rangle_\xi \mathcal{D}F_r^{(0)} \\ &-\frac{1}{4B_0^2}\Big\langle |b_\perp^{(0)}|^2 \partial_\xi b_\parallel^{(1)}\Big\rangle_\xi \mathcal{D}F_r^{(0)} - \frac{1}{4B_0}\partial_\tau\Big\langle |b_\perp^{(0)}|^2\Big\rangle_\xi \frac{\partial F_r^{(0)}}{\partial v_\perp} \\ &+\frac{1}{4B_0}\mathcal{D}\Big\langle \partial_\xi \overline{f}_r^{(1)} |b_\perp^{(0)}|^2\Big\rangle_\xi\Big] + \frac{v_\perp}{2}\Big[\Big\langle b_\perp^{(0)}\cdot\nabla_\perp \overline{f}_r^{(1)}\Big\rangle_\xi \\ &-\frac{1}{4B_0^2}\Big\langle |b_\perp^{(0)}|^2\partial_\xi b_\parallel^{(1)}\Big\rangle_\xi \mathcal{D}_1^-\mathcal{D}F_r^{(0)}\Big] + \frac{1}{4B_0}\partial_\tau\Big\langle |b_\perp^{(0)}|^2\Big\rangle_\xi \mathcal{D}F_r^{(0)}\Big\} \\ &+\frac{1}{2B_0}\Big[c\Big\langle e_\perp^{(4)}\cdot b_\perp^{(0)}\Big\rangle_\xi \frac{\partial F_r^{(0)}}{\partial v_\perp} + \Big\langle b_\perp^{(0)}\times b_\perp^{(4)}\Big\rangle_\xi\Big(-v_\parallel \frac{\partial F_r^{(0)}}{\partial v_\perp} + v_\perp \frac{\partial F_r^{(0)}}{\partial v_\parallel}\Big)\Big] \\ &+\frac{c}{2B_0^2}\Big\langle e_\parallel^{(3)}|b_\perp^{(0)}|^2\Big\rangle_\xi \frac{\partial}{\partial v_\parallel}\mathcal{D}F_r^{(0)} - \frac{1}{2B_0}\Big\langle b_\perp^{(0)}\times\partial_\xi^{-1}\partial_\tau b_\perp^{(0)}\frac{\partial \overline{f}_r^{(1)}}{\partial v_\perp}\Big\rangle_\xi \\ &+\frac{1}{2B_0}\Big\langle b_\perp^{(0)}\times b_\perp^{(2)}\mathcal{D}\overline{f}_r^{(1)}\Big\rangle_\xi + \frac{1}{4B_0}\Big(\frac{\partial}{\partial v_\perp} + \frac{2}{v_\perp}\Big)\Psi \\ &-\frac{1}{16B_0^3}\Big\langle |b_\perp^{(0)}|^2 (b_\perp^{(2)}\times b_\perp^{(0)})\Big\rangle_\xi \mathcal{D}_2^+\mathcal{D}_1^-\mathcal{D}F_r^{(0)} + \frac{m_r c}{4q_r B_0^3}(\lambda - v_\parallel)\Big\langle b_\parallel^{(1)}\partial_\xi |b_\perp^{(0)}|^2\Big\rangle_\xi \mathcal{D}F_r^{(0)} \\ &-\frac{1}{2B_0^2}\Big\langle b_\parallel^{(1)} b_\perp^{(0)}\times b_\perp^{(2)}\Big\rangle_\xi \mathcal{D}F_r^{(0)} + \frac{1}{2B_0^2}\Big\langle b_\parallel^{(1)} b_\perp^{(0)}\times \partial_\xi^{-1}\partial_\tau b_\perp^{(0)}\Big\rangle_\xi \frac{\partial F_r^{(0)}}{\partial v_\perp} \\ &-\frac{1}{4B_0}\mathcal{D}_2^+\Big\langle (b_y^{(0)2} - b_z^{(0)2})\langle \cos 2\phi \, \partial_\phi f_r^{(3)}\rangle + 2b_y^{(0)} b_z^{(0)}\langle \sin 2\phi \, \partial_\phi f_r^{(3)}\rangle\Big\rangle_\xi ,\end{aligned} \tag{109}$$

where the quantity Ψ need not to be explicited since it enters a term that does not contribute to the integral arising in the computation of the pressure tensor. One is then led to compute

$$\Big\langle (b_y^{(0)2} - b_z^{(0)2})\langle \cos 2\phi \, \partial_\phi f_r^{(3)}\rangle + 2b_y^{(0)} b_z^{(0)}\langle \sin 2\phi \, \partial_\phi f_r^{(3)}\rangle\Big\rangle_\xi$$

$$
\begin{aligned}
&= \frac{1}{4B_0\Omega_r}\Big\{\Big\langle \nabla_\perp\cdot(|b_\perp^{(0)}|^2 b_\perp^{(0)}) + 2|b_\perp^{(0)}|^2\partial_\xi b_\parallel^{(1)}\Big\rangle_\xi\Big\} v_\perp \mathcal{D}F_r^{(0)}\\
&-\frac{1}{4B_0^2}\Big\langle |b_\perp^{(0)}|^2 (b_\perp^{(0)}\times\partial_\xi^{-1}\partial_\tau b_\perp^{(0)})\Big\rangle_\xi \Big(\frac{\partial}{\partial v_\perp}-\frac{1}{v_\perp}\Big)\mathcal{D}F_r^{(0)}\\
&+\frac{1}{4B_0^2}\Big\langle |b_\perp^{(0)}|^2 b_\perp^{(0)}\times b_\perp^{(2)}\Big\rangle_\xi \mathcal{D}_1^- \mathcal{D}F_r^{(0)}\\
&+\frac{1}{2B_0}\mathcal{D}_1^-\Big\langle |b_\perp^{(0)}|^2\Big(b_y^{(0)}\Big\langle\cos\phi\,\partial_\phi f_r^{(2)}\rangle + b_z^{(0)}\langle\sin\phi\,\partial_\phi f_r^{(2)}\rangle\Big)\Big\rangle_\xi\\
&-\frac{1}{6B_0}\mathcal{D}_3^+\Big\langle (b_y^{(0)3}-3b_y^{(0)}b_z^{(0)2})\langle\cos 3\phi\,\partial_\phi f_r^{(2)}\rangle\\
&+(3b_y^{(0)2}b_z^{(0)}-b_z^{(0)3})\langle\sin 3\phi\partial_\phi f_r^{(2)}\rangle\Big\rangle_\xi
\end{aligned}
\tag{110}
$$

where

$$
\begin{aligned}
&\Big\langle |b_\perp^{(0)}|^2((b_y^{(0)}\langle\cos\phi\,\partial_\phi f_r^{(2)}\rangle + (b_z^{(0)}\langle\sin\phi\,\partial_\phi f_r^{(2)}\rangle)\Big\rangle_\xi =\\
&\frac{1}{2B_0}\Big\langle |b_\perp^{(0)}|^2 b_\perp^{(0)}\times b_\perp^{(2)}\Big\rangle_\xi \mathcal{D}F_r^{(0)} - \frac{1}{2B_0}\Big\langle |b_\perp^{(0)}|^2 b_\perp^{(0)}\times\partial_\xi^{-1}\partial_\tau b_\perp^{(0)}\Big\rangle_\xi \frac{\partial F_r^{(0)}}{\partial v_\perp}
\end{aligned}
\tag{111}
$$

and

$$
\Big\langle (b_y^{(0)3}-3b_y^{(0)}b_z^{(0)2})\langle\cos 3\phi\,\partial_\phi f_r^{(2)}\rangle + (3b_y^{(0)2}b_z^{(0)}-b_z^{(0)3})\langle\sin 3\phi\,\partial_\phi f_r^{(2)}\rangle\Big\rangle_\xi = 0. \tag{112}
$$

At this point, it is important to check that the system is closed in the sense that the quantities $b_\perp^{(2)}$ and $b_\perp^{(4)}$ do not enter the quantities of interest. Indeed, cancelations take place and one gets

$$
\begin{aligned}
&\frac{2}{c}\sum_r q_r n_r \sum_{p+q=4}\Big\langle b_\perp^{(p)}\cdot\int v_\perp v_\parallel\langle\vec{n}\frac{\partial f_r^{(q)}}{\partial\phi}\rangle\Big\rangle_\xi d^3v =\\
&\frac{2}{c}\sum_r q_r n_r\int v_\parallel v_\perp\Big\{-\frac{v_\parallel-\lambda}{4B_0\Omega_r}\frac{\partial F_r^{(0)}}{\partial v_\perp}\partial_\tau\Big\langle |b_\perp^{(0)}|^2\Big\rangle_\xi + \frac{v_\parallel-\lambda}{4B_0\Omega_r}\mathcal{D}\Big\langle\partial_\xi\overline{f}_r^{(1)}|b_\perp^{(0)}|^2\Big\rangle_\xi\\
&+\frac{v_\perp}{2\Omega_r}\Big[\Big\langle b_\perp^{(0)}\cdot\nabla_\perp\overline{f}_r^{(1)}\Big\rangle_\xi - \frac{1}{4B_0^2}\Big\langle |b_\perp^{(0)}|^2\partial_\xi b_\parallel^{(1)}\Big\rangle_\xi\mathcal{D}_1^-\mathcal{D}F_r^{(0)}\Big]\\
&+\frac{1}{4B_0\Omega_r}\Big\langle\partial_\tau|b_\perp^{(0)}|^2\Big\rangle_\xi\mathcal{D}F_r^{(0)} + \frac{c}{2B_0^2}\Big\langle e_\parallel^{(3)}|b_\perp^{(0)}|^2\Big\rangle_\xi\frac{\partial}{\partial v_\parallel}\mathcal{D}F_r^{(0)}\\
&-\frac{1}{2B_0}\Big\langle b_\perp^{(0)}\times\partial_\xi^{-1}\partial_\tau b_\perp^{(0)}\frac{\partial\overline{f}_r^{(1)}}{\partial v_\perp}\Big\rangle_\xi\\
&-\frac{1}{16B_0^2\Omega_r}\Big\langle\nabla_\perp\cdot(|b_\perp^{(0)}|^2 b_\perp^{(0)}) + 2|b_\perp^{(0)}|^2\partial_\xi b_\parallel^{(1)}\Big\rangle_\xi\Big]\mathcal{D}_2^+ v_\perp\mathcal{D}F_r^{(0)}\\
&+\frac{1}{16B_0^3}\Big\langle |b_\perp^{(0)}|^2\Big(b_\perp^{(0)}\times\partial_\xi^{-1}\partial_\tau b_\perp^{(0)}\Big)\Big\rangle_\xi\mathcal{D}_2^+\left(\frac{\partial}{\partial v_\perp}-\frac{1}{v_\perp}\right)\mathcal{D}F_r^{(0)}\\
&+\frac{1}{16B_0^3}\Big\langle |b_\perp^{(0)}|^2\Big(b_\perp^{(0)}\times\partial_\xi^{-1}\partial_\tau b_\perp^{(0)}\Big)\Big\rangle_\xi\mathcal{D}_2^+\mathcal{D}_1^-\frac{\partial F_r^{(0)}}{\partial v_\perp}\Big\}d^3v
\end{aligned}
\tag{113}
$$

where one may substitute $\Big\langle |b_\perp^{(0)}|^2\partial_\xi b_\parallel^{(1)}\Big\rangle_\xi = -\Big\langle |b_\perp^{(0)}|^2\nabla_\perp\cdot b_\perp^{(0)}\Big\rangle_\xi$.

We first note that in the above equation, no contribution originates from the terms involving the quantitiy $b_\perp^{(0)}\times\partial_\xi^{-1}\partial_\tau b_\perp^{(0)}$, because of the condition of quasi-neutrality that holds at the corresponding order. The observation is straightforward for the term containing $\overline{f}_r^{(1)}$. In the case of those involving $F_r^{(0)}$, a few integrals must be computed. One checks that $\int v_\perp v_\parallel\frac{\partial}{\partial v_\parallel}\mathcal{D}F_r^{(0)}d^3v = 2\lambda\int F_r^{(0)}d^3v$, $\int v_\perp v_\parallel\mathcal{D}_2^+\left(\frac{\partial}{\partial v_\perp}-\frac{1}{v_\perp}\right)F_r^{(0)}d^3v = -8\lambda\int F_r^{(0)}d^3v$ and $\int v_\perp v_\parallel\mathcal{D}_2^+\mathcal{D}_1^-\frac{\partial F_r^{(0)}}{\partial v_\perp} = -8\lambda\int F_r^{(0)}d^3v$. One then computes the coefficients

$$
\frac{2}{c}\sum_r\frac{1}{8B_0^2\Omega_r}q_r n_r\int v_\parallel v_\perp^2\mathcal{D}_1^-\mathcal{D}F_r^{(0)}d^3v = \frac{4\lambda}{B_0^3}(p_\perp^{(0)}-p_\parallel^{(0)}) \tag{114}
$$

$$\frac{2}{c}\sum_r \frac{1}{16B_0^2\Omega_r} q_r n_r \int v_\parallel v_\perp \mathcal{D}_2^+ v_\perp \mathcal{D} F_r^{(0)} d^3v = \frac{\lambda}{B_0^3} p_\perp^{(0)}. \tag{115}$$

Concerning the remaining contributions involving $\overline{f}_r^{(1)}$, one notices that only the fluctuations $\widetilde{\overline{f}}_r^{(1)}$ around the average value $\langle \overline{f}_r^{(1)} \rangle_\xi$ contribute. One writes

$$\begin{aligned}
&\frac{2}{c}\sum_r \frac{q_r n_r}{4B_0\Omega_r} \int v_\parallel v_\perp (v_\parallel - \lambda)\mathcal{D}\Big\langle \partial_\xi \overline{f}_r^{(1)} |b_\perp^{(0)}|^2 \Big\rangle_\xi \\
&\quad = \frac{1}{B_0^2}\sum_r m_r n_r \int [v_\parallel(v_\parallel - \lambda) - v_\perp^2)]\Big\langle (v_\parallel - \lambda)\partial_\xi \overline{f}_r^{(1)} |b_\perp^{(0)}|^2 \Big\rangle_\xi d^3v \\
&\quad -2\pi \frac{\lambda}{2B_0^2}\sum_r m_r n_r \int d(\frac{v_\perp^2}{2}) v_\perp^2 \Big\langle \Big(\int \partial_\xi \overline{f}_r^{(1)} dv_\parallel \Big) |b_\perp^{(0)}|^2 \Big\rangle_\xi.
\end{aligned} \tag{116}$$

One estimates the first term I_1 of the RHS of the above equation by means of eqs. (39)-(41), in the form

$$\begin{aligned}
I_1 &= \frac{1}{B_0^3}\sum_r m_r n_r \int [v_\parallel(v_\parallel - \lambda) - v_\perp^2)]\Big[-(v_\parallel - \lambda)\frac{v_\perp}{2}\frac{\partial F_r^{(0)}}{\partial v_\perp} + \frac{v_\perp^2}{2}\frac{\partial F_r^{(0)}}{\partial v_\parallel}\Big] d^3v \Big\langle \partial_\xi b_\parallel^{(1)} |b_\perp^{(0)}|^2 \Big\rangle_\xi \\
&= \frac{\lambda}{B_0^3}(5p_\perp^{(0)} - 2p_\parallel^{(0)})\Big\langle \partial_\xi b_\parallel^{(1)} |b_\perp^{(0)}|^2 \Big\rangle_\xi.
\end{aligned} \tag{117}$$

Using (52), one estimates the second term I_2 (without the minus sign) as

$$\begin{aligned}
I_2 &= \frac{2\lambda}{B_0^3} p_\perp^{(0)} \Big\langle |b_\perp^{(0)}|^2 \partial_\xi b_\parallel^{(1)} \Big\rangle_\xi + \frac{2\pi\lambda}{B_0^2}\sum_r m_r n_r \int d(\frac{v_\perp^2}{2})\frac{v_\perp^2}{2}\Big\langle |b_\perp^{(0)}|^2 \mathcal{G}_r \Big(\frac{v_\perp^2}{2} - \frac{q_r}{m_r}\mathcal{L}^{-1}\mathcal{M}\Big)\partial_\xi A \Big\rangle_\xi \\
&= \frac{2\lambda}{B_0^3} p_\perp^{(0)} \Big\langle |b_\perp^{(0)}|^2 \partial_\xi b_\parallel^{(1)} \Big\rangle_\xi + \frac{\lambda}{B_0^2}\Big\langle |b_\perp^{(0)}|^2 \Big(\mathcal{N} - \mathcal{M}^2\mathcal{L}^{-1}\Big)\partial_\xi A \Big\rangle_\xi.
\end{aligned} \tag{118}$$

One then has to estimate

$$\frac{2}{c}\sum_r \frac{1}{2\Omega_r} q_r n_r \int v_\perp^2 v_\parallel \widetilde{\overline{f}}_r^{(1)} d^3v = \frac{1}{B_0}\sum m_r n_r \Big[\int v_\perp^2 (v_\parallel - \lambda)\widetilde{\overline{f}}_r^{(1)} d^3v + \lambda \int v_\perp^2 \widetilde{\overline{f}}_r^{(1)} d^3v \Big]. \tag{119}$$

Using (39) and (40), the first term J_1 of the RHS is rewritten in the form

$$\begin{aligned}
J_1 &= \frac{1}{B_0}\sum_r m_r n_r \int v_\perp^2 (v_\parallel - \lambda)\Big[-\frac{v_\perp}{2}\frac{\partial F_r^{(0)}}{\partial v_\perp}\widetilde{A} + \frac{1}{2}\mathcal{D}_1^+ \mathcal{D} F_r^{(0)} \frac{\widetilde{|b_\perp^{(0)}|^2}}{2B_0^2}\Big] d^3v \\
&= -\frac{4\lambda}{B_0} p_\perp^0 \widetilde{A} + \frac{\lambda}{B_0}\left(-2\frac{B_0^2}{4\pi} + 6p_\perp^{(0)} - 4p_\parallel^{(0)} \right)\left(\frac{\widetilde{|b_\perp^{(0)}|^2}}{2B_0^2} \right)
\end{aligned} \tag{120}$$

and the second one J_2 is directly obtained from (55), as

$$J_2 = \frac{2\lambda}{B_0}\Big[\frac{B_0^2}{4\pi}\frac{\widetilde{|b_\perp^{(0)}|^2}}{2B_0^2} + \Big(2p_\perp^{(0)} + \mathcal{N} - \mathcal{M}^2\mathcal{L}^{-1}\Big)\widetilde{A}\Big]. \tag{121}$$

Putting together the above expansions, one obtains

$$\begin{aligned}
&\partial_\tau \sum_r m_r n_r \int \frac{v_\perp^2}{2}\langle \overline{f}_r^{(1)} \rangle_\xi \; d^3v = \Big(2p_\parallel^{(0)} - p_\perp^{(0)} - \frac{B_0^2}{4\pi}\Big)\partial_\tau \Big\langle \frac{|b_\perp^{(0)}|^2}{2B_0^2} \Big\rangle_\xi \\
&\quad +2p_\perp^{(0)}\partial_\tau \Big\langle \frac{b_\parallel^{(1)}}{B_0} \Big\rangle_\xi + \lambda\Big(\frac{B_0^2}{4\pi} + 2p_\perp^{(0)}\Big)\nabla_\perp \cdot \langle A \frac{b_\perp^{(0)}}{B_0} \rangle_\xi \\
&\quad +\frac{\lambda}{B_0}\nabla_\perp \cdot \Big\langle \frac{b_\perp^{(0)}}{B_0}(\mathcal{N} - \mathcal{M}^2\mathcal{L}^{-1})\widetilde{A} \Big\rangle_\xi - \lambda \Big\langle A(\mathcal{N} - \mathcal{M}^2\mathcal{L}^{-1})\partial_\xi A \Big\rangle_\xi
\end{aligned} \tag{122}$$

and

$$\partial_\tau \sum_r m_r n_r \int \frac{v_\parallel^2}{2} \langle \overline{f}_r^{(1)} \rangle_\xi \; d^3v = p_\parallel^{(0)} \partial_\tau \Big\langle \frac{b_\parallel^{(1)}}{B_0} \Big\rangle_\xi + (p_\perp^{(0)} - 2p_\parallel^{(0)}) \partial_\tau \Big\langle \frac{|b_\perp^{(0)}|^2}{B_0^2} \Big\rangle_\xi$$
$$+2\lambda \Big\langle A(\mathcal{N} - \mathcal{M}^2 \mathcal{L}^{-1}) \partial_\xi A \Big\rangle_\xi . \tag{123}$$

This leads to dynamical equations for the mean pressures. Using

$$\partial_\tau \Big\langle \frac{|b_\perp^{(0)}|^2}{2B_0^2} \Big\rangle_\xi = \frac{1}{2\lambda\rho^{(0)}} \Big[\nabla_\perp \cdot \Big\langle \widetilde{P} \frac{b_\perp^{(0)}}{B_0} \Big\rangle_\xi - \Big\langle \widetilde{A} (\mathcal{N} - \mathcal{M}^2 \mathcal{L}^{-1}) \partial_\xi \widetilde{A} \Big\rangle_\xi \Big], \tag{124}$$

one gets

$$\partial_\tau \Big(\frac{\langle p_\perp^{(1)} \rangle_\xi}{p_\perp^{(0)}} - \frac{\langle \rho^{(1)} \rangle_\xi}{\rho^{(0)}} - \langle A \rangle_\xi \Big) = 0. \tag{125}$$

For the mean parallel pressure, one has

$$\partial_\tau \Big(\frac{\langle p_\parallel^{(1)} \rangle_\xi}{p_\parallel^{(0)}} - 3 \frac{\langle \rho^{(1)} \rangle_\xi}{\rho^{(0)}} + 2\langle A \rangle_\xi \Big) = \frac{2\lambda}{p_\parallel^{(0)}} \Big\langle \widetilde{A} (\mathcal{N} - \mathcal{M}^2 \mathcal{L}^{-1}) \partial_\xi \widetilde{A} \Big\rangle_\xi . \tag{126}$$

E. Mean longitudinal magnetic perturbation

Using eq. (94) and eq. (125), one has from eq. (91)

$$\frac{d}{d\tau} \Gamma(\tau) = \partial_\tau \Big[\Big(\frac{B_0^2}{4\pi} + p_\perp^{(0)} \Big) \langle A \rangle_\xi + p_\perp^{(0)} \frac{\langle b_\parallel^{(1)} \rangle_\xi}{B_0} \Big] \tag{127}$$
$$= \partial_\tau \Big[\Big(\frac{B_0^2}{4\pi} + p_\perp^{(0)} \Big) \frac{\langle |b_\perp^{(0)}|^2 \rangle_\xi}{2B_0^2} + \Big(\frac{B_0^2}{4\pi} + 2p_\perp^{(0)} \Big) \frac{\langle \rho^{(1)} \rangle_\xi}{\rho^{(0)}} \Big]. \tag{128}$$

Averaging over the entire three-dimensional space and using the property of mass conservation to get rid of the density contribution, one obtains

$$\frac{d}{d\tau} \Gamma(\tau) = -\frac{1}{2\lambda\rho^{(0)}} \Big(\frac{B_0^2}{4\pi} + p_\perp^{(0)} \Big) \Big\langle \widetilde{A} (\mathcal{N} - \mathcal{M}^2 \mathcal{L}^{-1}) \partial_\xi \widetilde{A} \Big\rangle_{\xi,\eta,\zeta} \tag{129}$$

where $\langle \cdot \rangle_{\xi,\zeta,\eta}$ holds for the averaging on the full spatial domain. Substituting on the LHS of eq. (127), one gets

$$\frac{\langle b_\parallel^{(1)} \rangle_\xi}{B_0} = -\frac{2+\beta_\perp}{2(1+\beta_\perp)} \Big[\frac{\langle |b_\perp^{(0)}|^2 \rangle_\xi}{2B_0^2} + \frac{1}{\lambda\rho^{(0)}} \Big\langle \widetilde{A} (\mathcal{N} - \mathcal{M}^2 \mathcal{L}^{-1}) \partial_\xi \widetilde{A} \Big\rangle_{\xi,\eta,\zeta} \Big] \tag{130}$$

with $\beta_\perp = 8\pi p_\perp^{(0)}/B_0^2$. Note that this total pressure balance condition differs from that obtained in the case of a polytropic gas, not only by the presence of kinetic effects but also by the coefficient $(2+\beta_\perp)/2(1+\beta_\perp)$ instead of $1/(1+2\gamma^{-1}\beta)$.

F. Evolution of the mean field in the KDNLS equation

From eqs. (90) and (94), the mean field $\langle U \rangle_\xi$ entering eq. (63) satisfies

$$\partial_\tau \langle U \rangle_\xi = \partial_\tau \Big(\langle u_\parallel \rangle_\xi + \frac{\lambda}{2B_0} \langle b_\parallel^{(1)} \rangle_\xi + \frac{1}{\lambda\rho^{(0)}} (p_\parallel^{(0)} - p_\perp^{(0)}) \langle A \rangle_\xi + \frac{1}{2\lambda\rho^{(0)}} (\langle p_\perp^{(1)} \rangle_\xi - \langle p_\parallel^{(1)} \rangle_\xi) \Big). \tag{131}$$

Expressing the time derivatives arising in the RHS as obtained in the previous subsections, one finally gets

$$\partial_\tau \langle U \rangle_\xi = \frac{c_1}{\rho^{(0)}} \Big(\nabla_\perp \cdot \Big\langle \widetilde{P} \frac{b_\perp^{(0)}}{B_0} \Big\rangle_\xi - \Big\langle \widetilde{A} (\mathcal{N} - \mathcal{M}^2 \mathcal{L}^{-1}) \partial_\xi \widetilde{A} \Big\rangle_\xi \Big) - \frac{c_2}{\rho^{(0)}} \Big\langle \widetilde{A} (\mathcal{N} - \mathcal{M}^2 \mathcal{L}^{-1}) \partial_\xi \widetilde{A} \Big\rangle_{\xi,\eta,\zeta}, \tag{132}$$

with

$$c_1 = \frac{1}{2+\beta_\perp - \beta_\|}\Big(\frac{12+18\beta_\perp+5\beta_\perp^2}{8(1+\beta_\perp)}\Big) \quad , \quad c_2 = \frac{1}{2+\beta_\perp - \beta_\|}\Big(\frac{(2+\beta_\perp)^2}{8(1+\beta_\perp)}\Big) \tag{133}$$

where in addition to the previously defined parameter $\beta_\perp$, one introduced $\beta_\| = 8\pi p_\|^{(0)}/B_0^2$. Note that when kinetic effects are retained, a (time-dependent) convective velocity $\langle U\rangle_\xi$ is driven even in one space dimension.

V. CONCLUSION

Defining the operator $\mathcal{K} = \mathcal{N} - \mathcal{M}^2\mathcal{L}^{-1}$ [see eqs. (50,53) together with eq. (47)], one finally obtains that the dynamics of a long Alfvén wave train propagating parallel to an ambient field is governed by the closed system

$$\Big(\partial_\tau + \langle U\rangle_\xi \partial_\xi\Big) b_\perp^{(0)} + \frac{\partial}{\partial\xi}\Big(\frac{\widetilde{P} b_\perp^{(0)}}{2\lambda\rho^{(0)}}\Big) - \frac{B_0}{2\lambda\rho^{(0)}}\nabla_\perp \widetilde{P} + \delta\partial_{\xi\xi}\left(\widehat{x}\times b_\perp^{(0)}\right) = 0 \tag{134}$$

$$\rho^{(0)}\partial_\tau\langle U\rangle_\xi = c_1\left(\nabla_\perp\cdot\Big\langle \widetilde{P}\frac{b_\perp^{(0)}}{B_0}\Big\rangle_\xi - \Big\langle \widetilde{A}\mathcal{K}\partial_\xi\widetilde{A}\Big\rangle_\xi\right) - c_2\Big\langle\widetilde{A}\mathcal{K}\partial_\xi\widetilde{A}\Big\rangle_{\xi,\eta,\zeta}, \tag{135}$$

$$\partial_\xi \widetilde{b}_\|^{(1)} + \nabla_\perp\cdot b_\perp^{(0)} = 0, \tag{136}$$

with $\widetilde{A} = \dfrac{\widetilde{b}_\|^{(1)}}{B_0} + \dfrac{\widetilde{|b_\perp^{(0)}|^2}}{2B_0^2}$, $\widetilde{P} = \Big(\dfrac{B_0^2}{4\pi} + 2p_\perp^{(0)} + \mathcal{K}\Big)\widetilde{A}$ and the previously defined constants c_1 and c_2. One also establishes that the conservation of the energy (kinetic + thermal + magnetic) $E = \rho u^2/2 + b^2/8\pi + p_\perp + p_\|/2$ is recovered at the level of the asymptotic model.

Several problems can be addressed using the above multidimensional KDNLS system for wave trains. One of them concerns the influence of the Landau damping on the filamentation phenomenon.[14] In the Hall-MHD context, the envelope formalism predicts that (absolute) filamentation of long Alfvén waves requires the condition $\beta > 1$. Direct evidence of this effect was demonstrated by numerical integration of the three-dimensional DNLS equation with mean fields,[12] and also by comparison with direct numerical simulations of the Hall-MHD equations.[13]

Furthermore, the above system that provides an asymptotically exact description of long Alfvén waves, can also be used as a benchmark for Landau fluid models proposed to describe the large scale dynamics of a collisionless plasma permeated by a strong ambient field.[1] Such systems are constructed as fluid moment equations and can be viewed as a generalized magnetohydrodynamic description retaining linear Landau damping through a semi-heuristic approach.[1]

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APPENDIX A: CURRENT AND CHARGE BALANCES

Expanding eq. (3) to the successive orders, one obtains the various contributions to the electric current $j = \sum_r q_r n_r \int vF_r d^3v$, in the form $\sum_r q_r n_r \int vF_r^{(0)} d^3v = 0$, $\sum_r q_r n_r \int v f_r^{(0)} d^3v = 0$, $\sum_r q_r n_r \int v f_r^{(1)} d^3v = 0$. Non-zero contributions are obtained at the higher orders. One has

$$\frac{4\pi}{c}\sum_r q_r n_r \int \vec{v}_\perp f_r^{(2)} d^3v = \widehat{x}\times\partial_\xi b_\perp^{(0)} + \frac{\lambda}{c}\partial_\xi e_\perp^{(0)}. \tag{A.1}$$

Since $e_\perp^{(0)} = -\dfrac{\lambda}{c}\widehat{x}\times b_\perp^{(0)}$, the last term in the above equation, associated with the displacement current, is usually negligible. One gets

$$\frac{4\pi}{c}\sum_r q_r n_r \int (\widehat{x}\times\vec{v}_\perp)\, f_r^{(2)} d^3v = -\partial_\xi b_\perp^{(0)} \tag{A.2}$$

or, using the identity $\widehat{x} \times \vec{v}_\perp = \partial_\phi \vec{v}_\perp$,

$$\frac{4\pi}{c} \sum_r q_r n_r \int \vec{v}_\perp \frac{\partial f_r^{(2)}}{\partial \phi} d^3 v = \partial_\xi b_\perp^{(0)}. \tag{A.3}$$

The next two orders of the expansion give

$$\frac{4\pi}{c} \sum_r q_r n_r \int v_\parallel f_r^{(3)} d^3 v = \nabla_\perp \times b_\perp^{(0)} \tag{A.4}$$

and

$$\frac{4\pi}{c} \sum_r q_r n_r \int \vec{v}_\perp f_r^{(4)} d^3 v = \widehat{x} \times \partial_\xi b_\perp^{(2)} + (\nabla_\perp \times b_\parallel^{(1)} \widehat{x}) \tag{A.5}$$

or

$$\frac{4\pi}{c} \sum_r q_r n_r \int \vec{v}_\perp \frac{\partial f_r^{(4)}}{\partial \phi} d^3 v = \partial_\xi b_\perp^{(2)} - \widehat{x} \times (\nabla_\perp \times b_\parallel^{(1)} \widehat{x}) = \partial_\xi b_\perp^{(2)} - \nabla_\perp b_\parallel^{(1)}. \tag{A.6}$$

Similarly, expanding the Gauss equation (4) at the successive orders leads to the electric neutrality conditions $\sum_r q_r n_r = 0$ and $\sum_r q_r n_r \int \overline{f}_r^{(1)} d^3 v = 0$. Pushing further the expansion, it immediately follows that the condition of neutral locality is also obtained for the contribution arising from $f_r^{(2)}$. The first non-zero contribution reads $4\pi \sum_r q_r n_r \int f_r^{(3)} d^3 v = \nabla \cdot e_\perp^{(0)} = \frac{\lambda}{c} \nabla \times b_\perp^{(0)}$. Comparing with (A.1), one obtains $\displaystyle\sum_r q_r n_r \int f_r^{(3)} d^3 v = \left(\frac{\lambda}{c}\right) \frac{4\pi}{c} \sum_r q_r n_r \int v_\parallel f_r^{(3)} d^3 v$ that is negligible because of the factor $\frac{\lambda}{c}$.

APPENDIX B: AZIMUTHAL MODES OF THE DISTRIBUTION FUNCTION

The hierachy (6)-(12) implies that $F_r^{(0)}$ is independent of ϕ, $f_r^{(0)}$ involves only the mode $e^{i\phi}$, $f_r^{(1)}$ the modes $e^{2i\phi}$ and 1, $f_r^{(2)}$ the modes $e^{3i\phi}$ and $e^{i\phi}$, $f_r^{(3)}$ the modes $e^{4i\phi}$, $e^{2i\phi}$ and 1, $f_r^{(4)}$ the modes $e^{5i\phi}$, $e^{3i\phi}$ and $e^{i\phi}$ (together with their complex conjugates), etc. More generally, the $f_r^{(p)}$'s are obtained recursively in terms of the $f_r^{(q)}$'s with $q < p$. The computation of $\langle e^{i\phi} f_r^{(p)} \rangle$ where $\langle \cdot \rangle$ denotes the average on the ϕ variable involves the determination of various modes of $f_r^{(q)}$ with $q < p$. One is then led to establish the following estimates

$$\langle \cos\phi \, \frac{\partial}{\partial \phi} f_r^{(0)} \rangle = \frac{1}{2B_0} b_z^{(0)} \mathcal{D} F_r^{(0)} \tag{B.1}$$

$$\langle \sin\phi \, \frac{\partial}{\partial \phi} f_r^{(0)} \rangle = -\frac{1}{2B_0} b_y^{(0)} \mathcal{D} F_r^{(0)} \tag{B.2}$$

$$\langle \cos 2\phi \, \frac{\partial}{\partial \phi} f_r^{(1)} \rangle = \frac{1}{2B_0^2} b_y^{(0)} b_z^{(0)} \mathcal{D}_- \mathcal{D} F_r^{(0)} \tag{B.3}$$

$$\langle \sin 2\phi \, \frac{\partial}{\partial \phi} f_r^{(1)} \rangle = -\frac{1}{4B_0^2} (b_y^{(0)2} - b_z^{(0)2}) \mathcal{D}_- \mathcal{D} F_r^{(0)} \tag{B.4}$$

$$\begin{aligned} \langle \cos\phi \, \frac{\partial}{\partial \phi} f_r^{(2)} \rangle = \Big(&- \frac{m_r c}{2 q_r B_0^2} (\lambda - v_\parallel) \partial_\xi b_y^{(0)} + \frac{1}{2B_0} b_z^{(2)} - \frac{b_\parallel^{(1)}}{2B_0^2} b_z^{(0)} \Big) \mathcal{D} F_r^{(0)} \\ &- \frac{1}{2B_0} \partial_\xi^{-1} \partial_\tau b_z^{(0)} \frac{\partial F_r^{(0)}}{\partial v_\perp} - \frac{1}{16 B_0^3} |b_\perp^{(0)}|^2 b_z^{(0)} \mathcal{D}_2^+ \mathcal{D}_1^- \mathcal{D} F_r^{(0)} + \frac{1}{2B_0} b_z^{(0)} \mathcal{D} \overline{f}_r^{(1)} \end{aligned} \tag{B.5}$$

$$\begin{aligned} \langle \sin\phi \, \frac{\partial}{\partial \phi} f_r^{(2)} \rangle = \Big(&- \frac{m_r c}{2 q_r B_0^2} (\lambda - v_\parallel) \partial_\xi b_z^{(0)} - \frac{1}{2B_0} b_y^{(2)} + \frac{b_\parallel^{(1)}}{2B_0^2} b_y^{(0)} \Big) \mathcal{D} F_r^{(0)} \\ &+ \frac{1}{2B_0} \partial_\xi^{-1} \partial_\tau b_y^{(0)} \frac{\partial F_r^{(0)}}{\partial v_\perp} + \frac{1}{16 B_0^3} |b_\perp^{(0)}|^2 b_y^{(0)} \mathcal{D}_2^+ \mathcal{D}_1^- \mathcal{D} F_r^{(0)} - \frac{1}{2B_0} b_y^{(0)} \mathcal{D} \overline{f}_r^{(1)} \end{aligned} \tag{B.6}$$

$$\langle \cos 3\phi \, \frac{\partial}{\partial \phi} f_r^{(2)} \rangle = \frac{1}{16 B_0^3} \mathcal{D}_2^- \mathcal{D}_1^- \mathcal{D} F_r^{(0)} \Big(3 b_y^{(0)2} b_z^{(0)} - b_z^{(0)3} \Big) \tag{B.7}$$

$$\langle \sin 3\phi \, \frac{\partial}{\partial \phi} f_r^{(2)} \rangle = \frac{1}{16 B_0^3} \mathcal{D}_2^- \mathcal{D}_1^- \mathcal{D} F_r^{(0)} \Big(3 b_y^{(0)} b_z^{(0)2} - b_y^{(0)3} \Big) \tag{B.8}$$

$$\begin{aligned} \langle \cos 2\phi \, \frac{\partial}{\partial \phi} f_r^{(3)} \rangle = & \frac{1}{\Omega_r} \Big\{ \frac{1}{2} (\lambda - v_\parallel) \partial_\xi \Big\langle \sin 2\phi \frac{\partial}{\partial \phi} f_r^{(1)} \Big\rangle + \frac{v_\perp}{4 B_0} (\partial_y b_y^{(0)} - \partial_z b_z^{(0)}) \mathcal{D} F_r^{(0)} \Big\} \\ & - \frac{1}{4 B_0^2} \Big[b_y^{(0)} \partial_\xi^{-1} \partial_\tau b_z^{(0)} + b_z^{(0)} \partial_\xi^{-1} \partial_\tau b_y^{(0)} \Big] \Big(\frac{\partial}{\partial v_\perp} - \frac{1}{v_\perp} \Big) \mathcal{D} F_r^{(0)} \\ & + \frac{1}{4 B_0^2} \Big(b_y^{(2)} b_z^{(0)} + b_z^{(2)} b_y^{(0)} \Big) \mathcal{D}_1^- \mathcal{D} F_r^{(0)} - \frac{1}{B_0} b_\parallel^{(1)} \Big\langle \cos 2\phi \frac{\partial}{\partial \phi} f_r^{(1)} \Big\rangle \\ & + \frac{1}{2 B_0} \mathcal{D}_1^- \Big[b_y^{(0)} \Big\langle \cos \phi \frac{\partial}{\partial \phi} f_r^{(2)} \Big\rangle - b_z^{(0)} \Big\langle \sin \phi \frac{\partial}{\partial \phi} f_r^{(2)} \Big\rangle \Big] \\ & - \frac{1}{6 B_0} \mathcal{D}_3^+ \Big[b_y^{(0)} \Big\langle \cos 3\phi \frac{\partial}{\partial \phi} f_r^{(2)} \Big\rangle + b_z^{(0)} \Big\langle \sin 3\phi \frac{\partial}{\partial \phi} f_r^{(2)} \Big\rangle \Big] \end{aligned} \tag{B.9}$$

$$\begin{aligned} \langle \sin 2\phi \, \frac{\partial}{\partial \phi} f_r^{(3)} \rangle = & \frac{1}{\Omega_r} \Big\{ - \frac{1}{2} (\lambda - v_\parallel) \partial_\xi \Big\langle \cos 2\phi \frac{\partial}{\partial \phi} f_r^{(1)} \Big\rangle + \frac{v_\perp}{4 B_0} (\partial_y b_z^{(0)} + \partial_z b_y^{(0)}) \mathcal{D} F_r^{(0)} \Big\} \\ & + \frac{1}{4 B_0^2} \Big[b_y^{(0)} \partial_\xi^{-1} \partial_\tau b_y^{(0)} - b_z^{(0)} \partial_\xi^{-1} \partial_\tau b_z^{(0)} \Big] \Big(\frac{\partial}{\partial v_\perp} - \frac{1}{v_\perp} \Big) \mathcal{D} F_r^{(0)} \\ & - \frac{1}{4 B_0^2} \Big(b_y^{(2)} b_y^{(0)} - b_z^{(2)} b_z^{(0)} \Big) \mathcal{D}_1^- \mathcal{D} F_r^{(0)} - \frac{1}{B_0} b_\parallel^{(1)} \Big\langle \sin 2\phi \frac{\partial}{\partial \phi} f_r^{(1)} \Big\rangle \\ & + \frac{1}{2 B_0} \mathcal{D}_1^- \Big[b_y^{(0)} \Big\langle \sin \phi \frac{\partial}{\partial \phi} f_r^{(2)} \Big\rangle + b_z^{(0)} \Big\langle \cos \phi \frac{\partial}{\partial \phi} f_r^{(2)} \Big\rangle \Big] \\ & - \frac{1}{6 B_0} \mathcal{D}_3^+ \Big[b_y^{(0)} \Big\langle \sin 3\phi \frac{\partial}{\partial \phi} f_r^{(2)} \Big\rangle - b_z^{(0)} \Big\langle \cos 3\phi \frac{\partial}{\partial \phi} f_r^{(2)} \Big\rangle \Big]. \end{aligned} \tag{B.10}$$

$$\begin{aligned} \langle \cos \phi \, \frac{\partial}{\partial \phi} f_r^{(4)} \rangle = & \frac{1}{\Omega_r} \Big\{ - (v_\parallel - \lambda) \partial_\xi \Big\langle \sin \phi \; \partial_\phi f_r^{(2)} \Big\rangle \\ & + \frac{v_\perp}{2} \Big[\partial_y \overline{f}_r^{(1)} - \frac{1}{2} \partial_y \Big\langle \sin 2\phi \; \partial_\phi f_r^{(1)} \Big\rangle + \frac{1}{2} \partial_z \Big\langle \cos 2\phi \; \partial_\phi f_r^{(1)} \Big\rangle \Big] \\ & + \frac{1}{2 B_0} \partial_\tau b_y^{(0)} \mathcal{D} F_r^{(0)} \Big\} + \frac{1}{2 B_0} \Big[\Big(c e_y^{(4)} - v_\parallel b_z^{(4)} \Big) \frac{\partial}{\partial v_\perp} + v_\perp b_z^{(4)} \frac{\partial}{\partial v_\parallel} \Big] F_r^{(0)} \\ & + \frac{c}{2 B_0^2} e_\parallel^{(3)} b_y^{(0)} \frac{\partial}{\partial v_\parallel} \Big(\mathcal{D} F_r^{(0)} \Big) - \frac{1}{2 B_0^2} b_\parallel^{(3)} b_z^{(0)} \mathcal{D} F_r^{(0)} \\ & - \frac{1}{2 B_0} \partial_\xi^{-1} \partial_\tau b_z^{(0)} \frac{\partial \overline{f}_r^{(1)}}{\partial v_\perp} + \frac{1}{2 B_0} b_z^{(2)} \mathcal{D} \overline{f}_r^{(1)} \\ & + \frac{1}{4 B_0} \Big(\frac{\partial}{\partial v_\perp} + \frac{2}{v_\perp} \Big) \Big(\partial_\xi^{-1} \partial_\tau b_y^{(0)} \langle \cos 2\phi \; \partial_\phi f_r^{(1)} \rangle + \partial_\xi^{-1} \partial_\tau b_z^{(0)} \langle \sin 2\phi \; \partial_\phi f_r^{(1)} \rangle \Big) \\ & - \frac{1}{4 B_0} \mathcal{D}_2^+ \Big(b_y^{(2)} \langle \cos 2\phi \; \partial_\phi f_r^{(1)} \rangle + b_z^{(2)} \langle \sin 2\phi \; \partial_\phi f_r^{(1)} \rangle \Big) \\ & - \frac{1}{B_0} b_\parallel^{(1)} \langle \cos \phi \; \partial_\phi f_r^{(2)} \rangle + \frac{1}{2 B_0} b_z^{(0)} \mathcal{D} \overline{f}_r^{(3)} \\ & - \frac{1}{4 B_0} \mathcal{D}_2^+ \Big(b_y^{(0)} \langle \cos 2\phi \; \partial_\phi f_r^{(3)} \rangle + b_z^{(0)} \langle \sin 2\phi \; \partial_\phi f_r^{(3)} \rangle \Big) \end{aligned} \tag{B.11}$$

$$\begin{aligned} \langle \sin \phi \, \frac{\partial}{\partial \phi} f_r^{(4)} \rangle = & \frac{1}{\Omega_r} \Big\{ (v_\parallel - \lambda) \partial_\xi \Big\langle \cos \phi \; \partial_\phi f_r^{(2)} \Big\rangle \\ & + \frac{v_\perp}{2} \Big[\partial_z \overline{f}_r^{(1)} + \frac{1}{2} \partial_y \Big\langle \cos 2\phi \; \partial_\phi f_r^{(1)} \Big\rangle + \frac{1}{2} \partial_z \Big\langle \sin 2\phi \; \partial_\phi f_r^{(1)} \Big\rangle \Big] \\ & + \frac{1}{2 B_0} \partial_\tau b_z^{(0)} \mathcal{D} F_r^{(0)} \Big\} - \frac{1}{2 B_0} \Big[\Big(- c e_z^{(4)} - v_\parallel b_y^{(4)} \Big) \frac{\partial}{\partial v_\perp} + v_\perp b_y^{(4)} \frac{\partial}{\partial v_\parallel} \Big] F_r^{(0)} \end{aligned}$$

$$+\frac{c}{2B_0^2}e_\parallel^{(3)}b_z^{(0)}\frac{\partial}{\partial v_\parallel}\left(\mathcal{D}F_r^{(0)}\right)+\frac{1}{2B_0^2}b_\parallel^{(3)}b_y^{(0)}\mathcal{D}F_r^{(0)}$$
$$+\frac{1}{2B_0}\partial_\xi^{-1}\partial_\tau b_y^{(0)}\frac{\partial\overline{f}_r^{(1)}}{\partial v_\perp}-\frac{1}{2B_0}b_y^{(2)}\mathcal{D}\overline{f}_r^{(1)}$$
$$+\frac{1}{4B_0}\Big(\frac{\partial}{\partial v_\perp}+\frac{2}{v_\perp}\Big)\Big(\partial_\xi^{-1}\partial_\tau b_y^{(0)}\langle\sin 2\phi\ \partial_\phi f_r^{(1)}\rangle-\partial_\xi^{-1}\partial_\tau b_z^{(0)}\langle\cos 2\phi\ \partial_\phi f_r^{(1)}\rangle\Big)$$
$$-\frac{1}{4B_0}\mathcal{D}_2^+\Big(b_y^{(2)}\langle\sin 2\phi\ \partial_\phi f_r^{(1)}\rangle-b_z^{(2)}\langle\cos 2\phi\ \partial_\phi f_r^{(1)}\rangle\Big)$$
$$-\frac{1}{B_0}b_\parallel^{(1)}\langle\sin\phi\ \partial_\phi f_r^{(2)}\rangle-\frac{1}{2B_0}b_y^{(0)}\mathcal{D}\overline{f}_r^{(3)}$$
$$-\frac{1}{4B_0}\mathcal{D}_2^+\Big(b_y^{(0)}\langle\sin 2\phi\ \partial_\phi f_r^{(3)}\rangle-b_z^{(0)}\langle\cos 2\phi\ \partial_\phi f_r^{(3)}\rangle\Big)\tag{B.12}$$

Proof of equalities (B.1) and (B.2): These formulae are direct consequence of eq. (7) and the expression of $e^{(0)}$.
Proof of equalities (B.3) and (B.4): Equation (22) is rewritten

$$\partial_\phi f_r^{(1)}=\frac{1}{B_0^2}\Big[\frac{1}{2}(b_z^{(0)2}-b_y^{(0)2})\sin 2\phi+b_y^{(0)}b_z^{(0)}\cos 2\phi\Big]\mathcal{D}_1^-\mathcal{D}F_r^{(0)},\tag{B.13}$$

and the results follow directly.
Proof of equalities (B.5) and (B.6): The starting point is eq. (9). One then uses the relations

$$\Big\langle\cos\phi\Big(e^{(2)}+\frac{v\times b^{(2)}}{c}\Big)\cdot\nabla_v F_r^{(0)}\Big\rangle=\frac{1}{2c}\mathcal{D}F_r^{(0)}b_z^{(2)}-\frac{1}{2c}\partial_\xi^{-1}\partial_\tau b_z^{(2)}\frac{\partial F_r^{(0)}}{\partial v_\perp}\tag{B.14}$$
$$\Big\langle\sin\phi\Big(e^{(2)}+\frac{v\times b^{(2)}}{c}\Big)\cdot\nabla_v F_r^{(0)}\Big\rangle=-\frac{1}{2c}\mathcal{D}F_r^{(0)}b_y^{(2)}+\frac{1}{2c}\partial_\xi^{-1}\partial_\tau b_y^{(2)}\frac{\partial F_r^{(0)}}{\partial v_\perp}\tag{B.15}$$

and

$$\sum_{p+q=1}\Big(e^{(p)}+\frac{v\times b^{(p)}}{c}\Big)\cdot\nabla_v f_r^{(q)}=\frac{v_\parallel-\lambda}{c}\Big[\Big(b_y^{(0)}\sin\phi-b_z^{(0)}\cos\phi\Big)\frac{\partial f_r^{(1)}}{\partial v_\perp}+$$
$$\Big(b_y^{(0)}\cos\phi+b_z^{(0)}\sin\phi\Big)\frac{1}{v_\perp}\frac{\partial f_r^{(1)}}{\partial\phi}\Big]+\frac{v_\perp}{c}\Big(b_z^{(0)}\cos\phi-b_y^{(0)}\sin\phi\Big)\frac{\partial f_r^{(1)}}{\partial v_\parallel}-\frac{b_\parallel^{(1)}}{c}\frac{\partial f_r^{(0)}}{\partial\phi}.\tag{B.16}$$

The multiplication by $\cos\phi$ or $\sin\phi$ and the averaging on the ϕ variable lead to the introduction of the quantities $\langle\cos 2\phi\frac{\partial f_r^{(1)}}{\partial\phi}\rangle$ and $\langle\sin 2\phi\frac{\partial f_r^{(1)}}{\partial\phi}\rangle$ previously computed and also the mean value $\overline{f}_r^{(1)}=\langle f_r^{(1)}\rangle$. Simple algebra then leads to (B.5) and (B.6).
Proof of equalities (B.7) and (B.8): One writes

$$\langle\cos 3\phi\partial_\phi f_r^{(2)}\rangle=\frac{q_r}{m_r\Omega_r}\Big\langle\cos 3\phi\Big[\Big(e_\perp^{(0)}+\frac{v_\parallel\widehat{x}\times b_\perp^{(0)}}{c}\Big)\cdot\nabla_{v_\perp}f_r^{(1)}+\frac{\vec{v}_\perp\times b_\perp^{(0)}}{c}\frac{f_r^{(1)}}{\partial v_\parallel}\Big]\Big\rangle\tag{B.17}$$

and a similar expression for $\langle\sin 3\phi\partial_\phi f_r^{(2)}\rangle$. Expressing $\vec{v}_\perp$ and $\nabla_{v_\perp}$ in polar coordinates gives

$$\langle\cos 3\phi\ \partial_\phi f_r^{(2)}\rangle=\frac{1}{4B_0}\Big[\mathcal{D}_2^-\langle\cos 2\phi\ \partial_\phi f_r^{(1)}\rangle b_y^{(0)}-\mathcal{D}_2^-\langle\sin 2\phi\ \partial_\phi f_r^{(1)}\rangle b_z^{(0)}\Big]\tag{B.18}$$
$$\langle\sin 3\phi\ \partial_\phi f_r^{(2)}\rangle=\frac{1}{4B_0}\Big[\mathcal{D}_2^-\langle\sin 2\phi\ \partial_\phi f_r^{(1)}\rangle b_y^{(0)}+\mathcal{D}_2^-\langle\cos 2\phi\ \partial_\phi f_r^{(1)}\rangle b_z^{(0)}\Big],\tag{B.19}$$

which, by using (B.3) and (B.4), leads to (B.7) and (B.8).
Proof of equalities (B.9) and (B.10): Using eq. (10), one has to estimate

$$(v_\parallel-\lambda)\partial_\xi\langle\cos 2\phi f_r^{(1)}\rangle=-\frac{1}{2}(v_\parallel-\lambda)\partial_\xi\langle\sin 2\phi\ \partial_\phi f_r^{(1)}\rangle\tag{B.20}$$
$$(v_\parallel-\lambda)\partial_\xi\langle\sin 2\phi f_r^{(1)}\rangle=\frac{1}{2}(v_\parallel-\lambda)\partial_\xi\langle\cos 2\phi\ \partial_\phi f_r^{(1)}\rangle,\tag{B.21}$$

and

$$\langle \cos 2\phi (\vec{v}_\perp \cdot \nabla_\perp) f_r^{(0)} \rangle = \frac{v_\perp}{2} (\partial_y \langle \cos\phi f_r^{(0)} \rangle - \partial_z \langle \sin\phi f_r^{(0)} \rangle$$
$$= \frac{v_\perp}{4B_0} (\partial_y b_y^{(0)} - \partial_z b_z^{(0)}) \mathcal{D} F_r^{(0)} \quad \text{(B.22)}$$
$$\langle \sin 2\phi (\vec{v}_\perp \cdot \nabla_\perp) f_r^{(0)} \rangle = \frac{v_\perp}{4B_0} (\partial_y b_z^{(0)} + \partial_z b_y^{(0)}) \mathcal{D} F_r^{(0)}. \quad \text{(B.23)}$$

Denoting by $\alpha_\perp$ a transverse vector, one then has to compute various contributions of the form

$$\langle \cos 2\phi \ (\widehat{x} \times \alpha_\perp) \cdot \nabla_{v_\perp} f_r^{(0)} \rangle = -\frac{1}{2} \Big(\frac{\partial}{\partial v_\perp} - \frac{1}{v_\perp} \Big) \Big(\alpha_y \langle \cos\phi \ \partial_\phi f_r^{(0)} \rangle - \alpha_z \langle \sin\phi \ \partial_\phi f_r^{(0)} \rangle \Big)$$
$$= -\frac{1}{4B_0} (\alpha_y b_z^{(0)} + \alpha_z b_y^{(0)}) \Big(\frac{\partial}{\partial v_\perp} - \frac{1}{v_\perp} \Big) \mathcal{D} F_r^{(0)} \quad \text{(B.24)}$$
$$\langle \sin 2\phi \ (\widehat{x} \times \alpha_\perp) \cdot \nabla_{v_\perp} f_r^{(0)} \rangle = -\frac{1}{2} \Big(\frac{\partial}{\partial v_\perp} - \frac{1}{v_\perp} \Big) \Big(\alpha_y \langle \sin\phi \ \partial_\phi f_r^{(0)} \rangle + \alpha_z \langle \cos\phi \ \partial_\phi f_r^{(0)} \rangle \Big)$$
$$= \frac{1}{4B_0} (\alpha_y b_y^{(0)} - \alpha_z b_z^{(0)}) \Big(\frac{\partial}{\partial v_\perp} - \frac{1}{v_\perp} \Big) \mathcal{D} F_r^{(0)}, \quad \text{(B.25)}$$

together with

$$\Big\langle \cos 2\phi \frac{\vec{v}_\perp \times b_\perp^{(2)}}{c} \partial_{v_\parallel} f_r^{(0)} \Big\rangle = \frac{1}{4B_0 c} (b_y^{(0)} b_z^{(2)} + b_z^{(0)} b_y^{(2)}) v_\perp \partial_{v_\parallel} \mathcal{D} F_r^{(0)} \quad \text{(B.26)}$$
$$\Big\langle \sin 2\phi \frac{\vec{v}_\perp \times b_\perp^{(2)}}{c} \partial_{v_\parallel} f_r^{(0)} \Big\rangle = -\frac{1}{4B_0 c} (b_y^{(0)} b_y^{(2)} - b_z^{(0)} b_z^{(2)}) v_\perp \partial_{v_\parallel} \mathcal{D} F_r^{(0)} \quad \text{(B.27)}$$

and

$$\Big\langle \cos 2\phi \frac{\vec{v}_\perp \times b_\parallel^{(1)} \widehat{x}}{c} \nabla_{v_\perp} f_r^{(1)} \Big\rangle = -\frac{b_\parallel^{(1)}}{c} \Big\langle \cos 2\phi \ \partial_\phi f_r^{(1)} \Big\rangle \quad \text{(B.28)}$$
$$\Big\langle \sin 2\phi \frac{\vec{v}_\perp \times b_\parallel^{(1)} \widehat{x}}{c} \nabla_{v_\perp} f_r^{(1)} \Big\rangle = -\frac{b_\parallel^{(1)}}{c} \Big\langle \sin 2\phi \ \partial_\phi f_r^{(1)} \Big\rangle. \quad \text{(B.29)}$$

Furthermore,

$$\Big\langle \cos 2\phi \Big(e_\perp^{(0)} + \frac{v_\parallel \widehat{x} \times b_\perp^{(0)}}{c} \Big) \cdot \nabla_{v_\perp} \nabla_{v_\perp} f_r^{(2)} \Big\rangle =$$
$$\frac{\lambda - v_\parallel}{2c} \Big(\frac{\partial}{\partial v_\perp} - \frac{1}{v_\perp} \Big) \Big[b_y^{(0)} \langle \cos\phi \ \partial_\phi f_r^{(2)} \rangle - b_z^{(0)} \langle \sin\phi \ \partial_\phi f_r^{(2)} \rangle \Big]$$
$$-\frac{\lambda - v_\parallel}{6c} \Big(\frac{\partial}{\partial v_\perp} + \frac{3}{v_\perp} \Big) \Big[b_y^{(0)} \langle \cos 3\phi \ \partial_\phi f_r^{(2)} \rangle + b_z^{(0)} \langle \sin 3\phi \ \partial_\phi f_r^{(2)} \rangle \Big] \quad \text{(B.30)}$$
$$\langle \sin 2\phi \Big(e_\perp^{(0)} + \frac{v_\parallel \widehat{x} \times b_\perp^{(0)}}{c} \Big) \cdot \nabla_{v_\perp} f_r^{(2)} \rangle =$$
$$\frac{\lambda - v_\parallel}{2c} \Big(\frac{\partial}{\partial v_\perp} - \frac{1}{v_\perp} \Big) \Big[b_y^{(0)} \langle \sin\phi \ \partial_\phi f_r^{(2)} \rangle + b_z^{(0)} \langle \cos\phi \ \partial_\phi f_r^{(2)} \rangle \Big]$$
$$-\frac{\lambda - v_\parallel}{6c} \Big(\frac{\partial}{\partial v_\perp} + \frac{3}{v_\perp} \Big) \Big[b_y^{(0)} \langle \sin 3\phi \ \partial_\phi f_r^{(2)} \rangle - b_z^{(0)} \langle \cos 3\phi \ \partial_\phi f_r^{(2)} \rangle \Big] \quad \text{(B.31)}$$

and

$$\Big\langle \cos 2\phi \ \frac{\vec{v}_\perp \times b_\perp^{(0)}}{c} \frac{\partial f_r^{(2)}}{\partial v_\parallel} \Big\rangle = \frac{v_\perp}{2c} \frac{\partial}{\partial v_\parallel} \Big[\Big(b_y^{(0)} \langle \cos\phi \ \partial_\phi f_r^{(2)} \rangle - b_z^{(0)} \langle \sin\phi \ \partial_\phi f_r^{(2)} \rangle \Big)$$
$$-\frac{1}{3} \Big(b_y^{(0)} \langle \cos 3\phi \ \partial_\phi f_r^{(2)} \rangle + b_z^{(0)} \langle \sin 3\phi \ \partial_\phi f_r^{(2)} \rangle \Big) \Big] \quad \text{(B.32)}$$
$$\Big\langle \sin 2\phi \ \frac{\vec{v}_\perp \times b_\perp^{(0)}}{c} \frac{\partial f_r^{(2)}}{\partial v_\parallel} \Big\rangle = \frac{v_\perp}{2c} \frac{\partial}{\partial v_\parallel} \Big[\Big(b_y^{(0)} \langle \sin\phi \ \partial_\phi f_r^{(2)} \rangle + b_z^{(0)} \langle \cos\phi \ \partial_\phi f_r^{(2)} \rangle \Big)$$
$$-\frac{1}{3} \Big(b_y^{(0)} \langle \sin 3\phi \ \partial_\phi f_r^{(2)} \rangle - b_z^{(0)} \langle \cos 3\phi \ \partial_\phi f_r^{(2)} \rangle \Big) \Big] \quad \text{(B.33)}$$

Proof of equalities (B.11) and (B.12): One writes

$$\langle \cos\phi\ (v_\parallel - \lambda)\partial_\xi f_r^{(2)}\rangle = -(v_\parallel - \lambda)\partial_\xi \Big\langle \sin\phi\ \partial_\phi f_r^{(2)}\Big\rangle \tag{B.34}$$

$$\langle \sin\phi\ (v_\parallel - \lambda)\partial_\xi f_r^{(2)}\rangle = (v_\parallel - \lambda)\partial_\xi \Big\langle \cos\phi\ \partial_\phi f_r^{(2)}\Big\rangle, \tag{B.35}$$

$$\langle \cos\phi\ v_\perp\cdot\nabla_\perp f_r^{(1)}\rangle = \frac{v_\perp}{2}\Big[\partial_y \overline{f}_r^{(1)} - \frac{1}{2}\partial_y\Big\langle \sin 2\phi\ \partial_\phi f_r^{(1)}\Big\rangle + \frac{1}{2}\partial_z\Big\langle \cos 2\phi\ \partial_\phi f_r^{(1)}\Big\rangle\Big] \tag{B.36}$$

$$\langle \sin\phi\ v_\perp\cdot\nabla_\perp f_r^{(1)}\rangle = \frac{v_\perp}{2}\Big[\partial_z \overline{f}_r^{(1)} + \frac{1}{2}\partial_y\Big\langle \cos 2\phi\ \partial_\phi f_r^{(1)}\Big\rangle + \frac{1}{2}\partial_z\Big\langle \sin 2\phi\ \partial_\phi f_r^{(1)}\Big\rangle\Big] \tag{B.37}$$

$$\Big\langle \cos\phi\Big[\Big(e_\perp^{(4)} + \frac{v_\parallel\times b_\perp^{(4)}}{c}\Big)\cdot\nabla_{v_\perp}F_r^{(0)} + \frac{\vec{v}_\perp\times b_\perp^{(4)}}{c}\frac{\partial F_r^{(0)}}{\partial v_\parallel}\Big]\Big\rangle =$$
$$\frac{1}{2c}\Big[\Big(ce_y^{(4)} - v_\parallel b_z^{(4)}\Big)\frac{\partial}{\partial v_\perp} + v_\perp b_z^{(4)}\frac{\partial}{\partial v_\parallel}\Big]F_r^{(0)} \tag{B.38}$$

$$\Big\langle \sin\phi\Big[\Big(e_\perp^{(4)} + \frac{v_\parallel\times b_\perp^{(4)}}{c}\Big)\cdot\nabla_{v_\perp}F_r^{(0)} + \frac{\vec{v}_\perp\times b_\perp^{(4)}}{c}\frac{\partial F_r^{(0)}}{\partial v_\parallel}\Big]\Big\rangle =$$
$$-\frac{1}{2c}\Big[\Big(-ce_z^{(4)} - v_\parallel b_y^{(4)}\Big)\frac{\partial}{\partial v_\perp} + v_\perp b_y^{(4)}\frac{\partial}{\partial v_\parallel}\Big]F_r^{(0)} \tag{B.39}$$

$$\Big\langle \cos\phi\Big[\ e_\parallel^{(3)}\frac{\partial f_r^{(0)}}{\partial v_\parallel} + \frac{\vec{v}_\perp\times b_\parallel^{(3)}\widehat{x}}{c}\cdot\nabla_{v_\perp}f_r^{(0)}\Big]\Big\rangle$$
$$= \frac{1}{2B_0}e_\parallel^{(3)}b_y^{(0)}\frac{\partial}{\partial v_\parallel}\Big(\mathcal{D}F_r^{(0)}\Big) - \frac{1}{2cB_0}b_\parallel^{(3)}b_z^{(0)}\mathcal{D}F_r^{(0)} \tag{B.40}$$

$$\Big\langle \sin\phi\Big[\ e_\parallel^{(3)}\frac{\partial f_r^{(0)}}{\partial v_\parallel} + \frac{\vec{v}_\perp\times b_\parallel^{(3)}\widehat{x}}{c}\cdot\nabla_{v_\perp}f_r^{(0)}\Big]\Big\rangle$$
$$= \frac{1}{2B_0}e_\parallel^{(3)}b_z^{(0)}\frac{\partial}{\partial v_\parallel}\Big(\mathcal{D}F_r^{(0)}\Big) + \frac{1}{2cB_0}b_\parallel^{(3)}b_y^{(0)}\mathcal{D}F_r^{(0)} \tag{B.41}$$

$$\Big\langle \cos\phi\Big[\Big(e_\perp^{(2)} + \frac{v_\parallel\times b_\perp^{(2)}}{c}\Big)\cdot\nabla_{v_\perp}f_r^{(1)} + \frac{\vec{v}_\perp\times b_\perp^{(2)}}{c}\frac{\partial f_r^{(1)}}{\partial v_\parallel}\Big]\Big\rangle =$$
$$-\frac{1}{2c}\partial_\xi^{-1}\partial_\tau b_z^{(0)}\frac{\partial\overline{f}_r^{(1)}}{\partial v_\perp} + \frac{1}{2c}b_z^{(2)}\mathcal{D}\overline{f}_r^{(1)}$$
$$+\frac{1}{4c}\Big(\frac{\partial}{\partial v_\perp} + \frac{2}{v_\perp}\Big)\Big(\partial_\xi^{-1}\partial_\tau b_y^{(0)}\langle\cos 2\phi\ \partial_\phi f_r^{(1)}\rangle + \partial_\xi^{-1}\partial_\tau b_z^{(0)}\langle\sin 2\phi\ \partial_\phi f_r^{(1)}\rangle\Big)$$
$$-\frac{1}{4c}\mathcal{D}_2^+\Big(b_y^{(2)}\langle\cos 2\phi\ \partial_\phi f_r^{(1)}\rangle + b_z^{(2)}\langle\sin 2\phi\ \partial_\phi f_r^{(1)}\rangle\Big) \tag{B.42}$$

$$\Big\langle \sin\phi\Big[\Big(e_\perp^{(2)} + \frac{v_\parallel\times b_\perp^{(2)}}{c}\Big)\cdot\nabla_{v_\perp}f_r^{(1)} + \frac{\vec{v}_\perp\times b_\perp^{(2)}}{c}\frac{\partial f_r^{(1)}}{\partial v_\parallel}\Big]\Big\rangle =$$
$$\frac{1}{2c}\partial_\xi^{-1}\partial_\tau b_y^{(0)}\frac{\partial\overline{f}_r^{(1)}}{\partial v_\perp} - \frac{1}{2c}b_y^{(2)}\mathcal{D}\overline{f}_r^{(1)}$$
$$+\frac{1}{4c}\Big(\frac{\partial}{\partial v_\perp} + \frac{2}{v_\perp}\Big)\Big(\partial_\xi^{-1}\partial_\tau b_y^{(0)}\langle\sin 2\phi\ \partial_\phi f_r^{(1)}\rangle - \partial_\xi^{-1}\partial_\tau b_z^{(0)}\langle\cos 2\phi\ \partial_\phi f_r^{(1)}\rangle\Big)$$
$$-\frac{1}{4c}\mathcal{D}_2^+\Big(b_y^{(2)}\langle\sin 2\phi\ \partial_\phi f_r^{(1)}\rangle - b_z^{(2)}\langle\cos 2\phi\ \partial_\phi f_r^{(1)}\rangle\Big) \tag{B.43}$$

$$\Big\langle \cos\phi\frac{\vec{v}_\perp\times b_\parallel^{(1)}\widehat{x}}{c}\cdot\nabla_{v_\perp}f_r^{(2)}\Big\rangle = -\frac{b_\parallel^{(1)}}{c}\langle\cos\phi\partial_\phi f_r^{(2)}\rangle \tag{B.44}$$

$$\Big\langle \sin\phi\frac{\vec{v}_\perp\times b_\parallel^{(1)}\widehat{x}}{c}\cdot\nabla_{v_\perp}f_r^{(2)}\Big\rangle = -\frac{b_\parallel^{(1)}}{c}\langle\sin\phi\partial_\phi f_r^{(2)}\rangle \tag{B.45}$$

$$\Big\langle \cos\phi \Big[\Big(e_\perp^{(0)} + \frac{v_\parallel \times b_\perp^{(0)}}{c} \Big) \cdot \nabla_{v_\perp} f_r^{(3)} + \frac{\vec{v}_\perp \times b_\perp^{(0)}}{c} \frac{\partial f_r^{(3)}}{\partial v_\parallel} \Big] \Big\rangle = +\frac{1}{2c} b_z^{(0)} \mathcal{D} \overline{f}_r^{(3)}$$
$$-\frac{1}{4c} \mathcal{D}_2^+ \Big(b_y^{(0)} \langle \cos 2\phi \ \partial_\phi f_r^{(3)} \rangle + b_z^{(0)} \langle \sin 2\phi \ \partial_\phi f_r^{(3)} \rangle \Big) \quad \text{(B.46)}$$

$$\Big\langle \sin\phi \Big[\Big(e_\perp^{(2)} + \frac{v_\parallel \times b_\perp^{(0)}}{c} \Big) \cdot \nabla_{v_\perp} f_r^{(3)} + \frac{\vec{v}_\perp \times b_\perp^{(2)}}{c} \frac{\partial f_r^{(3)}}{\partial v_\parallel} \Big] \Big\rangle = -\frac{1}{2c} b_y^{(0)} \mathcal{D} \overline{f}_r^{(3)}$$
$$-\frac{1}{4c} \mathcal{D}_2^+ \Big(b_y^{(0)} \langle \sin 2\phi \ \partial_\phi f_r^{(3)} \rangle - b_z^{(0)} \langle \cos 2\phi \ \partial_\phi f_r^{(3)} \rangle \Big). \quad \text{(B.47)}$$

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