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CONVERGENCE OF A HIGH-ORDER SEMI-LAGRANGIAN SCHEME WITH PROPAGATION OF GRADIENTS FOR THE ONE-DIMENSIONAL VLASOV–POISSON SYSTEM*

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Abstract. In this paper we give a proof of convergence of a new numerical method introduced in [N. Besse and E. Sonnendrücker, *J. Comput. Phys.*, 191 (2003), pp. 341–376] for the Vlasov equation. The numerical method is based on the semi-Lagrangian principle and the transport of the gradient of the statistical distribution function in order to get a high-order and stable reconstruction. These kinds of new schemes have been successfully implemented on unstructured meshes of four-dimensional phase space (cf. [N. Besse, *Etude mathématique et numérique de l'equation de Vlasov sur des maillages non structurés de l'espace des phases*, thèse de l'Université Louis Pasteur, Strasbourg, France, 2003; N. Besse, J. Segré, and E. Sonnendrücker, *Transport Theory Statist. Phys.*, 34 (2005), pp. 311–332]). In order to make a rigorous proof of convergence of this method and simplify the convergence analysis, we have considered the periodic one-dimensional Vlasov–Poisson system in phase space on a grid. The distribution $f(t,x,v)$ and the electric field are shown to converge to the exact solution values in H^1 norm. The rate of convergence is of $\mathcal{O}(\Delta t^2 + \frac{\Delta x^{4-|\alpha|}}{\Delta t} + \frac{\Delta v^{4-|\alpha|}}{\Delta t})$, $\alpha \in \mathbb{N}^2,\ |\alpha| \le 1$.

Key words. Vlasov–Poisson system, semi-Lagrangian methods, high-order schemes, convergence analysis

AMS subject classifications. 65M12, 65M15, 82D10

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1. Introduction.

The numerical resolution of the Vlasov equation is most of the time performed by Lagrangian methods such as particle-in-cell methods which consist of approximating the plasma by a finite number of macroparticles (see Birdsall and Langdon [10] for more details). Although this method allows us to obtain satisfying results with a small number of particles, it is well known that the numerical noise inherent to the particle method becomes too significant to allow a precise description of the tail of the distribution function, which plays an important role in charged particle beams. To remedy this problem, Eulerian methods which consist in discretizing the Vlasov equation on a mesh of phase space have been proposed. One of these methods is semi-Lagrangian methods, which have efficiently been implemented using parallel computers [15] and give considerable promise in displaying the detailed structure of distribution functions in weak densities regions.

The author extends semi-Lagrangian schemes on unstructured meshes on two-, three-, and four-dimensional phase spaces with different kinds of high-order local interpolation operators and with the possibility of having a positive and conservative method by introducing a linear combination of low-order solution and high-order solution tempered by a limiter coefficient (cf. [5]). The interpolation operator considered involves the knowledge of the gradient of the distribution function which is obtained by solving a transport problem on the gradient or by differentiating the convected solution. For more details on how we get these schemes we refer the reader to [5, 8, 9].

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To simplify the convergence analysis we have considered this scheme on a uniform grid, without limiter coefficient, for the one-dimensional Vlasov–Poisson system.

Let us note that a first work on the convergence of one-dimensional particle methods is stated in [23], where Neunzert and Wick consider nonuniform initial loadings of particles asymptotically distributed with respect to initial data. Cottet and Raviart [14] present a mathematical analysis of the particle method for solving the one-dimensional Vlasov–Poisson system where uniform initial loadings of particles are considered. Then a lot of authors have studied the convergence of particle methods for the multidimensional Vlasov–Poisson system [18, 27, 28, 29]. They have also proved convergence results on random and deterministic particle methods for the Vlasov–Poisson–Fokker–Planck kinetic equations [21, 22]. Moreover Glassey and Schaeffer have done the convergence analysis of a particle method for the relativistic Vlasov–Maxwell system [20]. Schaeffer [24] has proved the convergence of a finite difference scheme for the one-dimensional Vlasov–Poisson–Fokker–Planck system, and Filbet [17] has shown the convergence of a finite volume scheme for the one-dimensional Vlasov–Poisson system. Finally the author [7] has proved the convergence of a semi-Lagrangian scheme on unstructured meshes (triangulation) of phase space in which a linear reconstruction is used, for the one-dimensional Vlasov–Poisson system. In the latter the author obtains a convergence rate in $\mathcal{O}(\Delta t^2 + h^2 + h^2/\Delta t)$ when $f \in \mathscr{C}_c^2([0,T] \times \mathbb{R}_x \times \mathbb{R}_v)$ (best convergence rate is in $\mathcal{O}(h^{4/3})$ when $\Delta t = h^{2/3}$) and a rate in $\mathcal{O}(\Delta t + h + h/\Delta t)$ when $f \in W_c^{1,\infty}([0,T] \times \mathbb{R}_x \times \mathbb{R}_v)$ (best convergence rate is in $\mathcal{O}(h^{1/2})$ when $\Delta t = h^{1/2}$).

Although a lot of papers present satisfactory numerical results using semi-Lagrangian methods [1, 2, 5, 13, 16, 25, 26], few rigorous mathematical results on convergence analysis of semi-Lagrangian methods have been stated. Despite interesting a priori estimates that have been pointed out (cf. [3, 4, 16]), a lot of work still remains to give complete and rigorous results in more general situations. The more difficult step in the convergence analysis of semi-Lagrangian methods is to obtain a stability result for the interpolation operators. If stability results in the $W^{1,\infty}$ norm seem inaccessible for high-order interpolation operators because of the Runge phenomena (artificial oscillations, whose amplitude increases with the degree of the polynomial in the case of Lagrange interpolation, appear at the edges of finite elements), a more appropriate mathematical framework is H^1 stability. If Fourier analysis tools such as Fourier series are useful for proving H^1 stability in case of grids, for unstructured meshes such as triangulation, convenient mathematical tools are lacking and have to be developed in the future. Nevertheless new results on convergence analysis of classes of high-order schemes can be found in [6].

This paper is organized as follows. In the first part we present the continuous problem. In the second part we expose the discrete problem and the numerical scheme to solve it. Then we study the convergence of our numerical scheme.

2. The continuous problem. Denoting by $f(t,x,v) \geq 0$ the distribution function of electrons in phase space (with mass normalized to one and the charge to plus one) and by $E(t,x)$ the self-consistent electric field, the adimensional Vlasov–Poisson system reads as

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + E(t,x)\frac{\partial f}{\partial v} = 0, \tag{2.1}$$

$$\frac{dE}{dx}(t,x) = \rho(t,x) = \int_{-\infty}^{+\infty} f(t,x,v)dv - 1, \tag{2.2}$$

where x and v are independent variables. We consider a periodic plasma of period L. Hence in (2.1) and (2.2) we have $x \in [0, L]$, $v \in \mathbb{R}$, $t \geq 0$, and the functions f and E satisfy the periodic boundary conditions

$$f(t, 0, v) = f(t, L, v), \qquad v \in \mathbb{R}, \quad t \geq 0, \tag{2.3}$$

and

$$E(t, 0) = E(t, L) \iff \frac{1}{L} \int_0^L \int_{-\infty}^{+\infty} f(t, x, v) dv dx = 1, \quad t \geq 0, \tag{2.4}$$

which means that the plasma is globally neutral. In order to have a well-posed problem we add to (2.1)–(2.4) a zero-mean electrostatic condition

$$\int_0^L E(t, x) dx = 0, \quad t \geq 0, \tag{2.5}$$

and an initial condition

$$f(0, x, v) = f_0(x, v), \quad x \in [0, L], \quad v \in \mathbb{R}. \tag{2.6}$$

Besides, by assuming that the electric field E is smooth enough, we can solve (2.1), (2.3), and (2.6) in the classical sense as follows. For the existence, the uniqueness, and the regularity of the solutions of the following differential system, we refer the reader to [11].

We consider the first-order differential system

$$\begin{aligned} \frac{dX}{dt}(t; s, x, v) &= V(t; s, x, v), \\ \frac{dV}{dt}(t; s, x, v) &= E(t, X(t; s, x, v)) \end{aligned} \tag{2.7}$$

and denote by $t \to (X(t; s, x, v), V(t; s, x, v))$ the characteristic curves, which are the solution of (2.7) with the initial condition

$$X(s; s, x, v) = x, \qquad V(s; s, x, v) = v. \tag{2.8}$$

Then the solution of problem (2.1), (2.6) is given by

$$f(t, x, v) = f_0(X(0; t, x, v), V(0; t, x, v)), \qquad x,\ v \in \mathbb{R}, \quad t \geq 0. \tag{2.9}$$

We note that the periodicity in x of $f_0(x, v)$ and $E(t, x)$ implies the periodicity in x of $f(t, x, v)$. Moreover as

$$\left| \frac{\partial(X, V)}{\partial(x, v)} \right| = 1,$$

we get

$$\frac{1}{L} \int_0^L \int_{-\infty}^{+\infty} f(t, x, v) dv dx = \frac{1}{L} \int_0^L \int_{-\infty}^{+\infty} f_0(x, v) dv dx = 1.$$

Therefore, according to the previous considerations, an equivalent form of the Vlasov–Poisson periodic problem is to find a couple (f, E), smooth enough, periodic with respect to x, with period L, and solving (2.2), (2.7), (2.8), and (2.9).

If we introduce the electrostatic potential $\phi = \phi(t,x)$ such that $E(t,x) = -\partial_x \phi(t,x)$, and if we denote by $G = G(x,y)$ the fundamental solution of the Laplacian operator in one dimension with a periodic boundary condition, therefore we obtain

$$E(t,x) = \int_0^L K(x,y) \left(\int_{-\infty}^{+\infty} f(t,y,v)dv - 1 \right) dy,$$

where

$$K(x,y) = -\partial_x G(x,y) = \begin{cases} \left(\dfrac{y}{L} - 1\right), & 0 \leq x < y, \\ \dfrac{y}{L}, & y < x \leq L. \end{cases}$$

2.1. Existence, uniqueness, and regularity of the solution of the continuous problem. In this section we recall a theorem of the existence of a classical solution for the Vlasov–Poisson system. The following theorem gives the existence, the uniqueness, and the regularity of the classical solutions, global in time, of the Vlasov–Poisson periodic system in one dimension. Let us note that a theory of weak solutions (in BV spaces) for the Vlasov equation has been recently developed in [12].

THEOREM 2.1. *Assuming* $f_0 \in \mathscr{C}^1_{c,per_x}(\mathbb{R}_x \times \mathbb{R}_v)$ *(continuously differentiable functions which are periodic with respect to* x *and compactly supported with respect to* v*), positive, periodic with respect to the variable* x *with period* L*, and* $Q(0) \leq R$*, with* $R > 0$ *and* $Q(t)$ *defined as*

$$Q(t) = 1 + \sup\left\{ |v| : \exists x \in [0,L], \quad \tau \in [0,t] \mid f(\tau,x,v) \neq 0 \right\},$$

and

$$\frac{1}{L} \int_0^L \int_{-\infty}^{+\infty} f_0(x,v) dv dx = 1,$$

then the periodic Vlasov–Poisson system has a unique classical solution (f,E) *periodic in* x*, with period* L*, for all time* t *in* $[0,T]$*, such that*

$$f \in \mathscr{C}^1_b\left(0,T;\mathscr{C}^1_{c,per_x}(\mathbb{R}_x \times \mathbb{R}_v)\right),$$

$$E \in \mathscr{C}^1_b\left(0,T;\mathscr{C}^1_{b,per_x}(\mathbb{R})\right),$$

and there exists a constant $C = C(R, f_0)$ *dependent on* R *and* f_0 *such that*

$$Q(T) \leq CT.$$

Moreover if we assume $f_0 \in \mathscr{C}^m_{c,per_x}(\mathbb{R}_x \times \mathbb{R}_v)$*, then* $(f,E) \in \mathscr{C}^m_b(0,T;\mathscr{C}^m_{c,per_x}(\mathbb{R}_x \times \mathbb{R}_v)) \times \mathscr{C}^m_b(0,T;\mathscr{C}^m_{b,per_x}(\mathbb{R}))$ *for all finite time* T*.*

Proof. For a proof we refer the reader to the treatise by Glassey [19]. ∎

3. The discrete problem. This section presents the description of the numerical scheme. Sections 3.1–3.3 are devoted to some notation and definitions (definitions of the interpolation operators and the transport operators in order to discretize in time and space the nonlinear transport equations, on a fixed phase-space grid). Section 3.4 resumes the algorithm in a simple way and refers the reader to the previous subsections for more details.

3.1. Interpolation operator. Let $\Omega = [0,L] \times [-R,R]$, with $R > Q(T)$, and let $\mathcal{M}_h$ be a Cartesian mesh of the phase space Ω. The grid $\mathcal{M}_h$ is given by a first increasing sequence $(x_i)_{i\in\{0,\dots,N_x\}}$ of the interval $[0,L]$ and a second increasing sequence $(v_i)_{i\in\{0,\dots,N_v\}}$ of the interval $[-R,R]$.

Let $\Delta x_i = x_{i+1} - x_i$ be the physical space set and $\Delta v_i = v_{i+1} - v_i$ be the velocity space set. In order to simplify the convergence analysis we suppose that $\Delta x_i = \Delta x$ and $\Delta v_i = \Delta v$. We call h a generic discretization parameter which stands for Δx or Δv. We define the one-dimensional Hermite interpolation operator $\mathcal{I}_h^{\mathcal{H}}$ as

$$\mathcal{I}_h^{\mathcal{H}} f(z)_{|_{[z_i,z_{i+1}]}} = f(z_i)\phi_i(z) + f(z_{i+1})\phi_{i+1}(z) + \dot f(z_i)\psi_i(z) + \dot f(z_{i+1})\psi_{i+1}(z),$$

where

$$\phi_i(z) = \frac{(z-z_{i+1})^2[(z_i - z_{i+1}) - 2(z-z_i)]}{(z_i - z_{i+1})^3}, \qquad \psi_i(z) = \frac{(z-z_i)(z-z_{i+1})^2}{(z_i-z_{i+1})^2},$$

$$\phi_{i+1}(z) = \frac{(z-z_i)^2[(z_i - z_{i+1}) + 2(z-z_{i+1})]}{(z_i - z_{i+1})^3}, \qquad \psi_{i+1}(z) = \frac{(z-z_i)^2(z-z_{i+1})}{(z_i-z_{i+1})^2},$$

and the one-dimensional Lagrange interpolation operator of degree three $\mathcal{I}_h^{\mathcal{L}}$ as

$$\mathcal{I}_h^{\mathcal{L}} f(z)_{|_{[z_i,z_{i+1}]}} = \sum_{k=i-1}^{i+2} f(z_k)\ell_k^3(z), \quad \text{where } \ell_k^3(z) = \prod_{\substack{i=k-1\\ i\neq k}}^{i=k+2} \frac{(z-z_i)}{(z_k - z_i)}.$$

Let $\delta(y,y_i) = 1$ if $y = y_i$ and zero otherwise; then we consider $\mathcal{H}_h$ the interpolation operator defined by $\mathcal{H}_h f(x,v) = \sum_{j=0}^{N_v} \mathcal{I}_h^{\mathcal{H}} f(x,v_j)\delta(v,v_j)$ for advection in the x-direction and $\mathcal{H}_h f(x,v) = \sum_{i=0}^{N_x} \mathcal{I}_h^{\mathcal{H}} f(x_i,v)\delta(x,x_i)$ for advection in the v-direction. In the same way we define $\mathcal{L}_h$ thanks to $\mathcal{I}_h^{\mathcal{L}}$.

3.2. Transport operators. In this section we introduce some transport operators. Let $\mathbb{S}_{h,\xi}$ be the translation operator defined by $\mathbb{S}_{h,\xi} f(t,x,v) = f(t,x-h,v-\xi)$. We suppose that $t \in [t^n, t^{n+1}]$, and then we define $\mathcal{T}_1^{0,0}$, $\mathcal{T}_2^{0,0}$, $\mathcal{T}_1^{1,0}$, $\mathcal{T}_1^{0,1}$, $\mathcal{T}_2^{1,0}$, and $\mathcal{T}_2^{0,1}$ as follows:

$$\begin{aligned}
\mathcal{T}_1^{0,0} f(t,x,v) &= f(t, x - v\Delta t/2, v),\\
\mathcal{T}_2^{0,0} f(t,x,v) &= f\left(t, x, v - \Delta t E(t^{n+1/2}, x)\right),\\
\mathcal{T}_1^{1,0} f(t,x,v) &= \partial_x \mathcal{T}_1^{0,0} f(t,x,v)\\
&= \partial_x \left(f(t, x - v\Delta t/2, v)\right) = \partial_x f(t, x - v\Delta t/2, v),\\
\mathcal{T}_1^{0,1} f(t,x,v) &= \partial_v \mathcal{T}_1^{0,0} f(t,x,v)\\
&= \partial_v \left(f(t, x - v\Delta t/2, v)\right)\\
&= \partial_v f(t, x - v\Delta t/2, v) - \frac{\Delta t}{2}\partial_x f(t, x - v\Delta t/2, v),\\
\mathcal{T}_2^{1,0} f(t,x,v) &= \partial_x \mathcal{T}_2^{0,0} f(t,x,v)\\
&= \partial_x \left(f\left(t, x, v - \Delta t E(t^{n+1/2}, x)\right)\right)\\
&= \partial_x f\left(t, x, v - \Delta t E(t^{n+1/2}, x)\right)\\
&\quad - \Delta t \partial_x E\left(t^{n+1/2}, x\right) \partial_v f\left(t, x, v - \Delta t E(t^{n+1/2}, x)\right),
\end{aligned}$$

$$\begin{aligned}\mathcal{T}_2^{0,1} f(t,x,v) &= \partial_v \mathcal{T}_2^{0,0} f(t,x,v) \\ &= \partial_v \left(f\left(t,x,v-\Delta t E(t^{n+1/2},x)\right)\right) = \partial_v f\left(t,x,v-\Delta t E(t^{n+1/2},x)\right),\end{aligned}$$

where $E(t,x)$ is the solution of the Poisson problem (2.2) and (2.5). For a given couple (x_k, v_l) there exists a real $\theta_k \in]0,1[$ and an integer m with $(k+m,l) \in \mathcal{M}_h$ such that $x_k - v_l \Delta t/2 = x_{k+m} + \theta_k \Delta x$. We set

$$\phi_0(z) = \frac{1}{h^3}(h+2z)(h-z)^2, \qquad \phi_1(z) = \frac{z^2}{h^3}(3h-2z),$$
$$\psi_0(z) = \frac{z}{h^2}(h-z)^2, \qquad \psi_1(z) = \frac{z^2}{h^2}(z-h),$$

where h denotes Δx or Δv. Therefore, using the notation $f_{k,l}(t) = f(t,x_k,v_l)$ and $\partial_x f_{k,l}(t) = (\partial_x f)(t,x_k,v_l)$, we define $\widetilde{\mathcal{T}}_1^{0,0}$, $\widetilde{\mathcal{T}}_1^{1,0}$, and $\widetilde{\mathcal{T}}_1^{0,1}$ as follows:

$$\begin{aligned}\left(\widetilde{\mathcal{T}}_1^{0,0} f(t)\right)_{k,l} &= \left(\mathbb{S}_{v\Delta t/2,0}\mathcal{H}_h f\right)(t,x_k,v_l) \\ &= f_{m+k,l}(t)\phi_0(\theta_k\Delta x) + f_{m+k+1,l}(t)\phi_1(\theta_k\Delta x) \\ &\quad + \partial_x f_{m+k,l}(t)\psi_0(\theta_k\Delta x) + \partial_x f_{m+k+1,l}(t)\psi_1(\theta_k\Delta x), \\ \left(\widetilde{\mathcal{T}}_1^{1,0} f(t)\right)_{k,l} &= \left(\mathbb{S}_{v\Delta t/2,0}\partial_x\mathcal{H}_h f\right)(t,x_k,v_l), \\ &= f_{m+k,l}(t)\dot{\phi}_0(\theta_k\Delta x) + f_{m+k+1,l}(t)\dot{\phi}_1(\theta_k\Delta x) \\ &\quad + \partial_x f_{m+k,l}(t)\dot{\psi}_0(\theta_k\Delta x) + \partial_x f_{m+k+1,l}(t)\dot{\psi}_1(\theta_k\Delta x), \\ \left(\widetilde{\mathcal{T}}_1^{0,1} f(t)\right)_{k,l} &= \left(\mathbb{S}_{v\Delta t/2,0}\mathcal{L}_h\partial_v f\right)(t,x_k,v_l) - \frac{\Delta t}{2}\left(\mathbb{S}_{v\Delta t/2,0}\partial_x\mathcal{H}_h f\right)(t,x_k,v_l) \\ &= \sum_{i=k-1}^{k+2} \partial_v f_{m+i,l}(t)\ell_i^3((m+i+\theta_k)\Delta x)) \\ &\quad - \frac{\Delta t}{2}\{f_{m+k,l}(t)\dot{\phi}_0(\theta_k\Delta x) + f_{m+k+1,l}(t)\dot{\phi}_1(\theta_k\Delta x) \\ &\quad + \partial_x f_{m+k,l}(t)\dot{\psi}_0(\theta_k\Delta x) + \partial_x f_{m+k+1,l}(t)\dot{\psi}_1(\theta_k\Delta x)\}.\end{aligned}$$

Here we have used the Lagrange interpolation of degree three on the gradient in order to keep high-order accuracy.

Then we introduce the following two Poisson problems:

$$\text{(P)} \qquad \left\{ \left(\frac{d}{dx}E_h^{n+1/2}\right)(x) = \int_v \widetilde{\mathcal{T}}_1^{0,0} f_h(t^n)dv - 1, \quad \int_0^L E_h^{n+1/2}(x)dx = 0 \right.$$

and

$$\text{(P}'\text{)} \left\{ \left(\frac{d}{dx}\left(\frac{d}{dx}E_h^{n+1/2}\right)\right)(x) = \int_v \widetilde{\mathcal{T}}_1^{1,0} f_h(t^n)dv, \quad \int_0^L \left(\frac{d}{dx}E_h^{n+1/2}\right)(x)dx = 0. \right.$$

In order to discretize (P) and (P′) we use a finite difference formula of order four for the x-derivative and a quadrature formula of order four for the integration in v. As a consequence we have the following two schemes:

$$
(\mathrm{P}_h)\begin{cases}\dfrac{1}{12\Delta x}\{8(E_h^{n+1/2}(x_{k+1})-E_h^{n+1/2}(x_{k-1}))-(E_h^{n+1/2}(x_{k+2})-E_h^{n+1/2}(x_{k-2}))\}\\ =-1+\dfrac{\Delta v}{3}\Bigg\{\left(\widetilde{\mathcal{T}}_1^{0,0}f_h(t^n)\right)_{k,0}+\left(\widetilde{\mathcal{T}}_1^{0,0}f_h(t^n)\right)_{k,N_v}\\ +4\displaystyle\sum_{l,\mathrm{odd}}\left(\widetilde{\mathcal{T}}_1^{0,0}f_h(t^n)\right)_{k,l}+2\sum_{l,\mathrm{even}}\left(\widetilde{\mathcal{T}}_1^{0,0}f_h(t^n)\right)_{k,l}\Bigg\},\\ E_h^{n+1/2}(x_0)+E_h^{n+1/2}(x_{N_x})+4\displaystyle\sum_{l,\mathrm{odd}}E_h^{n+1/2}(x_l)+2\sum_{l,\mathrm{even}}E_h^{n+1/2}(x_l)=0,\end{cases}
$$

$$
(\mathrm{P}'_h)\begin{cases}\dfrac{1}{12\Delta x}\{8(\partial_xE_h^{n+1/2}(x_{k+1})-\partial_xE_h^{n+1/2}(x_{k-1}))-(\partial_xE_h^{n+1/2}(x_{k+2})\\ -\partial_xE_h^{n+1/2}(x_{k-2}))\}=\dfrac{\Delta v}{3}\Bigg\{\left(\widetilde{\mathcal{T}}_1^{1,0}f_h(t^n)\right)_{k,0}+\left(\widetilde{\mathcal{T}}_1^{1,0}f_h(t^n)\right)_{k,N_v}\\ +4\displaystyle\sum_{l,\mathrm{odd}}\left(\widetilde{\mathcal{T}}_1^{1,0}f_h(t^n)\right)_{k,l}+2\sum_{l,\mathrm{even}}\left(\widetilde{\mathcal{T}}_1^{1,0}f_h(t^n)\right)_{k,l}\Bigg\},\\ \partial_xE_h^{n+1/2}(x_0)+\partial_xE_h^{n+1/2}(x_{N_x})+4\displaystyle\sum_{l,\mathrm{odd}}\partial_xE_h^{n+1/2}(x_l)+2\sum_{l,\mathrm{even}}\partial_xE_h^{n+1/2}(x_l)=0.\end{cases}
$$

As we choose periodic boundary conditions in the x-direction we have

$$
E_h^{n+1/2}(x_{l+N_x+1})=E_h^{n+1/2}(x_l),\quad \partial_xE_h^{n+1/2}(x_{l+N_x+1})=\partial_xE_h^{n+1/2}(x_l),\quad l\in\mathbb{Z}.
$$

Finally we build $E_h^{n+1/2}(x)$ by using the Hermite interpolation operator $\mathcal{I}_h^{\mathcal{H}}$, and, with $x=x_i+\theta\Delta x$, $0\le\theta\le 1$, we get

$$
\begin{aligned}E_h^{n+1/2}(x)_{|_{[x_i,x_{i+1}]}}&=E_h^{n+1/2}(x_i)\phi_0(\theta\Delta x)+E_h^{n+1/2}(x_{i+1})\phi_1(\theta\Delta x)\\ &\quad+\partial_xE_h^{n+1/2}(x_i)\psi_0(\theta\Delta x)+\partial_xE_h^{n+1/2}(x_{i+1})\psi_1(\theta\Delta x).\end{aligned}
$$

For a given couple (x_k,v_l) there exists a real $\theta_l\in\,]0,1[$ and an integer m with $(k,l+m)\in\mathcal{M}_h$ such that $v_l-\Delta tE_h^{n+1/2}(x_k)=v_{l+m}+\theta_l\Delta v$. Therefore, using the notation $\partial_vf_{k,l}(t)=(\partial_vf)(t,x_k,v_l)$, we consider the transport operators $\widehat{\mathcal{T}}_2^{0,0}$, $\widehat{\mathcal{T}}_2^{1,0}$, and $\widehat{\mathcal{T}}_2^{0,1}$, defined by

$$
\begin{aligned}\widehat{\mathcal{T}}_2^{0,0}f(t,x,v)&=f\left(t,x,v-\Delta tE_h^{n+1/2}(x)\right),\\ \widehat{\mathcal{T}}_2^{1,0}f(t,x,v)&=\partial_x\widehat{\mathcal{T}}_2^{0,0}f(t,x,v)\\ &=\partial_x\left(f\left(t,x,v-\Delta tE_h^{n+1/2}(x)\right)\right)\\ &=\partial_xf\left(t,x,v-\Delta tE_h^{n+1/2}(x)\right)\\ &\quad-\Delta t\partial_xE_h^{n+1/2}(x)\partial_vf\left(t,x,v-\Delta tE_h^{n+1/2}(x)\right),\\ \widehat{\mathcal{T}}_2^{0,1}f(t,x,v)&=\partial_v\widehat{\mathcal{T}}_2^{0,0}f(t,x,v)\\ &=\partial_v\left(f\left(t,x,v-\Delta tE_h^{n+1/2}(x)\right)\right)=\partial_vf\left(t,x,v-\Delta tE_h^{n+1/2}(x)\right),\end{aligned}
$$

and $\widetilde{\mathcal{T}}_2^{0,0}$, $\widetilde{\mathcal{T}}_2^{1,0}$, and $\widetilde{\mathcal{T}}_2^{0,1}$, defined by

$$\begin{aligned}
\left(\widetilde{\mathcal{T}}_2^{0,0} f(t)\right)_{k,l} &= \left(\mathbb{S}_{0,\Delta t E_h} \mathcal{H}_h f\right)(t, x_k, v_l) \\
&= f_{k,m+l}(t)\phi_0(\theta_l \Delta v) + f_{k,m+l+1}(t)\phi_1(\theta_l \Delta v) \\
&+ \partial_v f_{k,m+l}(t)\psi_0(\theta_l \Delta v) + \partial_v f_{k,m+l+1}(t)\psi_1(\theta_l \Delta v), \\
\left(\widetilde{\mathcal{T}}_2^{1,0} f(t)\right)_{k,l} &= \left(\mathbb{S}_{0,\Delta t E_h} \mathcal{L}_h \partial_x f\right)(t, x_k, v_l) - \Delta t \partial_x E_h^{n+1/2}(x_k)\left(\mathbb{S}_{0,\Delta t E_h} \partial_v \mathcal{H}_h f\right)(t, x_k, v_l) \\
&= \sum_{i=l-1}^{l+2} \partial_x f_{k,m+i}(t) \ell_i^3((m+i+\theta_l)\Delta v) \\
&- \Delta t \partial_x E_h^{n+1/2}(x_k)\{\partial_v f_{k,m+l}(t)\dot{\phi}_0(\theta_l \Delta v) + f_{k,m+l+1}(t)\dot{\phi}_1(\theta_l \Delta v) \\
&+ \partial_v f_{k,m+l}(t)\dot{\psi}_0(\theta_l \Delta v) + \partial_v f_{k,m+l+1}(t)\dot{\psi}_1(\theta_l \Delta v)\}, \\
\left(\widetilde{\mathcal{T}}_2^{0,1} f(t)\right)_{k,l} &= \left(\mathbb{S}_{0,\Delta t E_h} \partial_v \mathcal{H}_h f\right)(t, x_k, v_l) \\
&= f_{k,m+l}(t)\dot{\phi}_0(\theta_l \Delta v) + f_{k,m+l+1}(t)\dot{\phi}_1(\theta_l \Delta v) \\
&+ \partial_v f_{k,m+l}(t)\dot{\psi}_0(\theta_l \Delta v) + \partial_v f_{k,m+l+1}(t)\dot{\psi}_1(\theta_l \Delta v).
\end{aligned}$$

Let us notice that the transport operators $\widetilde{\mathcal{T}}_1^\alpha$ and $\widetilde{\mathcal{T}}_2^\alpha$ act only on a Cartesian grid function and that the operators $\mathcal{T}_1^\alpha$, $\mathcal{T}_2^\alpha$, and $\widehat{\mathcal{T}}_2^\alpha$ act only on functions defined in a Cartesian coordinate system. Therefore when we make the composition of these operators we must take into account this constraint. In order to be more precise we give an abstract example of the composition of such an operator. Let φ and ψ be two applications from $\mathbb{R}^2$ into $\mathbb{R}^2$:

$$\varphi: \quad \begin{aligned} \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x,v) &\longrightarrow \varphi(x,v) = (\varphi_1(x,v), \varphi_2(x,v)) \end{aligned} \qquad \Bigg| \qquad \psi: \quad \begin{aligned} \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x,v) &\longrightarrow \psi(x,v) \\ &= (\psi_1(x,v), \psi_2(x,v)). \end{aligned}$$

Let Γ_1 and Γ_1 be two transport operators which act only on functions defined in a Cartesian coordinate system. The operators Γ_1 and Γ_2 are defined as follows:

$$\Gamma_1 f(x,v) = f(\varphi_1(x,v), \varphi_2(x,v)) \quad \text{and} \quad \Gamma_2 f(x,v) = f(\psi_1(x,v), \psi_2(x,v)).$$

Let us now compute $\Gamma_1 \circ \Gamma_2 \circ \Gamma_1 f(x,v)$. First by applying Γ_1 to f we get

$$\Gamma_1 \circ \Gamma_2 \circ \Gamma_1 f(x,v) = \Gamma_1 \circ \Gamma_2 f(\varphi_1(x,v), \varphi_2(x,v)).$$

Now the function $f(\cdot,\cdot)$ is no longer defined in a Cartesian coordinate system, but it is defined in a curvilign coordinate system given by the mapping φ. Nevertheless there exists a function g defined in the original Cartesian coordinate system (x,v) such that $g(x,v) = f(\varphi_1(x,v), \varphi_2(x,v))$. We can now apply the operator Γ_2 to g, and we obtain

$$\begin{aligned}
\Gamma_1 \circ \Gamma_2 \circ \Gamma_1 f(x,v) &= \Gamma_1 \circ \Gamma_2 g(x,v) \\
&= \Gamma_1 g(\psi_1(x,v), \psi_2(x,v)) \\
&= \Gamma_1 f(\varphi_1(\psi_1(x,v), \psi_2(x,v)), \varphi_2(\psi_1(x,v), \psi_2(x,v))).
\end{aligned}$$

Following the same argument as before, there exists a function h which is defined in the original Cartesian coordinate system (x,v) such that $h(x,v) = g(\psi_1(x,v), \psi_2(x,v))$.

Therefore we get

$$
\begin{aligned}
\Gamma_1 \circ \Gamma_2 \circ \Gamma_1 f(x,v) &= \Gamma_1 \circ \Gamma_2 g(x,v) \ = \ \Gamma_1 g(\psi_1(x,v),\psi_2(x,v)) \ = \ \Gamma_1 h(x,v) \\
&= h(\varphi_1(x,v),\varphi_2(x,v)) \\
&= g(\psi_1(\varphi_1(x,v),\varphi_2(x,v)),\psi_2(\varphi_1(x,v),\varphi_2(x,v))) \\
&= f\left(\varphi_1(\psi_1(\varphi_1(x,v),\varphi_2(x,v)),\psi_2(\varphi_1(x,v),\varphi_2(x,v))),\right. \\
&\qquad \left.\varphi_2(\psi_1(\varphi_1(x,v),\varphi_2(x,v)),\psi_2(\varphi_1(x,v),\varphi_2(x,v)))\right).
\end{aligned}
$$

3.3. Notation and definitions. In this section we introduce some notation and definitions. The grid function $f = \{f_{i,j}\}_{(i,j)\in\mathcal{M}_h}$ belongs to functional space $L^2_h(\Omega)$ if the norm $\|\cdot\|_{L^2_h(\Omega)}$ defined as

$$
\|f\|_{L^2_h(\Omega)} = \left(\Delta x \Delta v \sum_{(i,j)\in\mathcal{M}_h} |f_{i,j}|^2\right)^{1/2}
$$

is uniformly bounded.

Let

$$
\begin{aligned}
G: \quad \mathcal{M}_h \quad &\longrightarrow \quad \mathbb{R}^2, \\
(i,j) \quad &\longrightarrow \quad (G_{1_{i,j}}, G_{2_{i,j}}),
\end{aligned}
$$

with $G_1,\ G_2 \in L^2_h(\Omega)$. Then we define the norm $\|\cdot\|_{\mathbb{L}^2_h(\Omega)}$ as

$$
\|G\|^2_{\mathbb{L}^2_h(\Omega)} = \|G_1\|^2_{L^2_h(\Omega)} + \|G_2\|^2_{L^2_h(\Omega)}.
$$

Let $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ be the vectors $(\mu_0,\dots,\mu_j,\dots,\mu_{N_v})$ and $(\nu_0,\dots,\nu_i,\dots,\nu_{N_x})$, with $0 \le \mu_j, \nu_i \le 1$ for all $(i,j) \in [0,N_x]\times[0,N_v]$. Then we define the norm $\|\cdot\|_{L^2_h,\Delta_h^{\boldsymbol{\mu},\boldsymbol{\nu}}}$ by

$$
\|f\|_{L^2_h,\Delta_h^{\boldsymbol{\mu},\boldsymbol{\nu}}} = \left(\Delta x \Delta v \sum_{(i,j)\in\mathcal{M}_h} |f_{i+\mu_j,j+\nu_i}|^2\right)^{1/2}.
$$

We also define the translation operator $\tau_{\boldsymbol{\mu},\boldsymbol{\nu}}$ as follows:

$$
(\tau_{\boldsymbol{\mu},\boldsymbol{\nu}} f)_{i,j} = f_{i+\mu_j,j+\nu_i} = f(x_i + \mu_j \Delta x, v_j + \nu_i \Delta v).
$$

We continue by introducing the space $\ell^\infty(0,T;X)$ defined by

$$
\begin{aligned}
\ell^\infty(0,T;X) := \Big\{ f : \{t^0,\dots,t^M\} \to X | M\Delta t = T, \\
||f||_{\ell^\infty(0,T;X)} = \max_{1\le n\le M} ||f(t^n)||_X < \infty \Big\},
\end{aligned}
$$

where X denotes a functional space with norm $||.||_X$, and the norm $\|\cdot\|_{L_h^\infty}$ defined by

$$
\|f\|_{L_h^\infty} = \max_{(i,j)\in\mathcal{M}_h} |f_{i,j}|.
$$

We introduce now the notation

$$
\mathcal{T}^\alpha, \quad \mathcal{T} \in \left\{\mathcal{T}_1, \mathcal{T}_2, \widetilde{\mathcal{T}}_1, \widetilde{\mathcal{T}}_2, \widehat{\mathcal{T}}_2\right\},
$$

where $\alpha = (\alpha_1,\alpha_2) \in \{(0,0),(1,0),(0,1)\}$, with $|\alpha| = \alpha_1 + \alpha_2$. In order to simplify the notation the following conventions are used:

$$
\nabla \mathcal{T} = \begin{pmatrix} \mathcal{T}^{1,0} \\ \mathcal{T}^{0,1} \end{pmatrix}, \qquad \partial^\alpha f \in \{\partial^{0,0} f, \partial^{1,0} f, \partial^{0,1} f\} = \{f, \partial_x f, \partial_v f\}.
$$

3.4. The numerical scheme. We suppose that we know $f_h(t^n) = \{f^n_{i,j}, \partial_x f^n_{i,j}, \partial_v f^n_{i,j}\}_{(i,j)\in\mathcal{M}_h}$. Therefore the numerical scheme which allows us to go from time t^n to t^{n+1} and compute $f_h(t^{n+1}) = \{f^{n+1}_{i,j}, \partial_x f^{n+1}_{i,j}, \partial_v f^{n+1}_{i,j}\}_{(i,j)\in\mathcal{M}_h}$ can be described in four steps:

(A1) We evaluate the distribution function and its partial derivatives at time t^n at the foot of the field-free characteristics starting at $\{(x_k, v_l)\}_{(k,l)\in\mathcal{M}_h}$ at time $t^{n+1/2}$ using $\mathcal{L}_h$ and $\mathcal{H}_h$, the Lagrange and Hermite interpolation operators, respectively. Then we obtain three new grid functions: One approximates the particle distribution, and the other two approximate its gradients. These actions are described by the transport operators $\widetilde{\mathcal{T}}_1^\alpha$. The superscripts denote the position or velocity gradient under consideration.

(A2) The output from (A1) is integrated with respect to velocity (by a quadrature formula of order four) to provide an approximation for the density and its gradient at time $t^{n+1/2}$, which are then substituted into the Poisson equations (P_h) and (P'_h) to compute the approximation of the electric field and its gradient at time $t^{n+1/2}$, respectively.

(A3) Therefore the result obtained from (A1) is evaluated at the foot of the velocity characteristic starting at $\{(x_k, v_l)\}_{(k,l)\in\mathcal{M}_h}$ at time t^{n+1} with the acceleration field found in (A2) using $\mathcal{L}_h$ and $\mathcal{H}_h$, the Lagrange and Hermite interpolation operators, respectively. Then we obtain three new grid functions: One approximates the particle distribution, and the other two approximate its gradients. These actions are described by the transport operators $\widetilde{\mathcal{T}}_2^\alpha$, with the superscripts denoting the appropriate gradient under consideration.

(A4) Between time $t^{n+1/2}$ and t^{n+1}, we apply step (A1) to the output from (A3). This action is described by the transport operators $\widetilde{\mathcal{T}}_1^\alpha$. Then we obtain $f_h(t^{n+1})$, the distribution function and its gradients at time t^{n+1}, which are the new initial data for the algorithm (A1)–(A4).

By using transport operators defined in section 3.2 the numerical scheme can be written as

$$\left(\partial^\alpha f_h(t^{n+1})\right)_{k,l} = \left(\widetilde{\mathcal{T}}_1^\alpha \circ \widetilde{\mathcal{T}}_2^\alpha \circ \widetilde{\mathcal{T}}_1^\alpha f_h(t^n)\right)_{k,l}, \quad |\alpha| \le 1,\ k \in [0, N_x],\ l \in [0, N_v],$$

with $(\partial^\alpha f_h^0) = (\partial^\alpha f_0)_{k,l}$, $|\alpha| \le 1$, as a discretization of the initial data f_0, $\partial^\alpha f_h^n(x_0 + L, v_l) = \partial^\alpha f_h^n(x_0, v_l)$, $|\alpha| \le 1$, for all $l \in [0, N_v]$, the boundary condition in the x-direction, and $\partial^\alpha f_h^n(x, v_l) = 0$, for all $|v_l| > R$, for all $x \in [0, L]$, $|\alpha| \le 1$, the boundary condition in the v-direction.

Let us note that the method presented in this paper belongs to the class of semi-Lagrangian methods but has the distinctive feature of using a splitting à la Strang for the operators associated to the transport of the distribution function and its gradient.

4. Convergence analysis. Let us state the convergence theorem and give the a priori error estimates.

THEOREM 4.1. *Assume that $f \in \mathscr{C}^2(0, T; \mathscr{C}^4_{c,per_x}(\mathbb{R}_x \times \mathbb{R}_v))$, is positive, and is periodic with respect to the variable x with period L; then the numerical solution of the Vlasov–Poisson system (f_h, E_h), computed by the numerical scheme exposed in section 3.4, converges towards the solution (f, E) of the periodic Vlasov–Poisson system, and there exists a constant $C = C(||f||_{\mathscr{C}^2(0,T;\mathscr{C}^4_b(\Omega))})$ independent of Δt, Δx,* *and Δv such that for $|\alpha| \leq 1$ we have*

$$||\partial^\alpha f - \partial^\alpha f_h||_{l^\infty\left(0,T;L^2_h(\Omega)\right)} \leq C\Bigg(\Delta t^2 + \frac{\Delta x^{4-|\alpha|} + \Delta v^{4-|\alpha|}}{\Delta t}. + \frac{\left(\Delta x^4 + \Delta v^4\right)}{\Delta x^{|\alpha|}}\left(1 + \frac{\Delta x}{\Delta t} + \frac{1}{\Delta t}\right)\Bigg)$$

and

$$||\partial^\alpha E - \partial^\alpha E_h||_{l^\infty\left(0,T;L^\infty_h([0,L])\right)} \leq C\Bigg(\Delta t^2 + \Delta x^4 + \Delta v^4 + \Delta x^{4-|\alpha|} + \frac{\left(\Delta x^4 + \Delta v^4\right)}{\Delta x^{|\alpha|}}\left(1 + \frac{\Delta x}{\Delta t} + \frac{1}{\Delta t}\right) + \frac{\Delta x^{4-|\alpha|} + \Delta v^{4-|\alpha|}}{\Delta t}\Bigg).$$

Remark 4.2. Theorem 4.1 tells us that the convergence occurred with a rate in $\mathcal{O}(\Delta t^2 + \Delta t^{-1}(\Delta x^{4-|\alpha|} + \Delta v^{4-|\alpha|}))$, $\alpha \in \mathbb{N}^2$, $|\alpha| \leq 1$. If $f \in \mathscr{C}^1(0,T;\mathscr{C}^2_{c,per_x}(\mathbb{R}_x \times \mathbb{R}_v))$, the convergence rate will be in $\mathcal{O}(\Delta t + \Delta t^{-1}(\Delta x^{2-|\alpha|} + \Delta v^{2-|\alpha|}))$, $\alpha \in \mathbb{N}^2$, $|\alpha| \leq 1$.

Idea of the proof. We want to evaluate the global error at the time t^{n+1} in the L^2_h norm

$$\|\partial^\alpha e^{n+1}\|_{L^2_h(\Omega)} = ||\partial^\alpha f(t^{n+1}) - \partial^\alpha f_h(t^{n+1})||_{L^2_h(\Omega)}.$$

Therefore we decompose $\partial^\alpha f(t^{n+1}, x_k, v_l) - \partial^\alpha f_h(t^{n+1}, x_k, v_l)$ as follows:

$$\begin{aligned}
&\partial^\alpha f(t^{n+1}, x_k, v_l) - \partial^\alpha f_h(t^{n+1}, x_k, v_l)\\
\text{(I1)}\quad &= \partial^\alpha f(t^{n+1}, x_k, v_l) - (\mathcal{T}_1^\alpha \circ \mathcal{T}_2^\alpha \circ \mathcal{T}_1^\alpha f(t^n))_{k,l}\\
\text{(I2)}\quad &+ (\mathcal{T}_1^\alpha \circ \mathcal{T}_2^\alpha \circ \mathcal{T}_1^\alpha f(t^n))_{k,l} - \left(\mathcal{T}_1^\alpha \circ \widehat{\mathcal{T}}_2^\alpha \circ \mathcal{T}_1^\alpha f(t^n)\right)_{k,l}\\
\text{(I3)}\quad &+ \left(\mathcal{T}_1^\alpha \circ \widehat{\mathcal{T}}_2^\alpha \circ \mathcal{T}_1^\alpha f(t^n)\right)_{k,l} - \left(\widetilde{\mathcal{T}}_1^\alpha \circ \widetilde{\mathcal{T}}_2^\alpha \circ \widetilde{\mathcal{T}}_1^\alpha f(t^n)\right)_{k,l}\\
\text{(I4)}\quad &+ \left(\widetilde{\mathcal{T}}_1^\alpha \circ \widetilde{\mathcal{T}}_2^\alpha \circ \widetilde{\mathcal{T}}_1^\alpha (f(t^n) - f_h(t^n))\right)_{k,l}.
\end{aligned}$$

In order to estimate $\|\partial^\alpha e^{n+1}\|_{L^2_h(\Omega)}$ we will estimate the four right-hand terms of the previous equation. These estimates are described in the following section.

A priori estimates. The following lemma gives an estimate for the term (I1), which can be viewed as a discretization error in time.

Lemma 4.3. *Assuming that $f \in \mathscr{C}^2_b(0,T;\mathscr{C}^2_{c,per_x}(\mathbb{R}_x \times \mathbb{R}_v))$, then there exists a constant C such that for $|\alpha| \leq 1$ we get*

$$\left\|\partial^\alpha f(t^{n+1}) - \mathcal{T}_1^\alpha \circ \mathcal{T}_2^\alpha \circ \mathcal{T}_1^\alpha f(t^n)\right\|_{L^2_h(\Omega)} \leq C\Delta t^3.$$

Proof. On one hand we can write

$$\begin{aligned}
f(t^{n+1}, x, v) &= f(t^{n+1}, X(t^{n+1}; t^{n+1}, x, v), V(t^{n+1}; t^{n+1}, x, v))\\
&= f(t^n, X(t^n; t^{n+1}, x, v), V(t^n; t^{n+1}, x, v))\\
&= f(t^n, X(t^n), V(t^n))
\end{aligned}$$

and

$$\nabla f(t^{n+1},x,v) = \begin{pmatrix} \dfrac{\partial f}{\partial x}(t^{n+1},x,v) \\ \dfrac{\partial f}{\partial v}(t^{n+1},x,v) \end{pmatrix}$$
$$= \begin{pmatrix} \dfrac{\partial X}{\partial x}(t^n) & \dfrac{\partial V}{\partial x}(t^n) \\ \dfrac{\partial X}{\partial v}(t^n) & \dfrac{\partial V}{\partial v}(t^n) \end{pmatrix} \begin{pmatrix} \dfrac{\partial f}{\partial X}(t^n, X(t^n), V(t^n)) \\ \dfrac{\partial f}{\partial V}(t^n, X(t^n), V(t^n)) \end{pmatrix}$$
$$= \nabla(X(t^n), V(t^n))\nabla f(t^n, X(t^n), V(t^n)).$$

Now we introduce $\mathbb{T}_1$ and $\mathbb{T}_2$ defined by

$$\mathbb{T}_1 = \begin{pmatrix} \mathcal{T}_1^{0,0} \\ \mathcal{T}_1^{1,0} \\ \mathcal{T}_1^{0,1} \end{pmatrix}, \qquad \mathbb{T}_2 = \begin{pmatrix} \mathcal{T}_2^{0,0} \\ \mathcal{T}_2^{1,0} \\ \mathcal{T}_2^{0,1} \end{pmatrix}.$$

On the other hand we get

$$\mathbb{T}_1 \circ \mathbb{T}_2 \circ \mathbb{T}_1 f(t^n) = \mathbb{T}_1 \circ \mathbb{T}_2 \circ \mathbb{T}_1 f(t^n, x, v)$$
$$= \mathbb{T}_1 \circ \mathbb{T}_2 \begin{pmatrix} f(t^n, x - v\Delta t/2, v) \\ \partial_x f(t^n, x - v\Delta t/2, v) \\ \partial_v f(t^n, x - v\Delta t/2, v) - \frac{\Delta t}{2}\partial_x f(t^n, x - v\Delta t/2, v) \end{pmatrix}$$
$$= \mathbb{T}_1 \begin{pmatrix} f(t^n, x - v\frac{\Delta t}{2} + \frac{\Delta t^2}{2}E(t^{n+1/2},x), v - \Delta t E(t^{n+1/2},x)) \\ \\ \partial_x f(t^n, x - v\frac{\Delta t}{2} + \frac{\Delta t^2}{2}E(t^{n+1/2},x), v - \Delta t E(t^{n+1/2},x)) \\ -\Delta t \partial_x E(t^{n+1/2},x) \\ \times\{\partial_v f(t^n, x - v\frac{\Delta t}{2} + \frac{\Delta t^2}{2}E(t^{n+1/2},x), v - \Delta t E(t^{n+1/2},x)) \\ -\frac{\Delta t}{2}\partial_x f(t^n, x - v\frac{\Delta t}{2} + \frac{\Delta t^2}{2}E(t^{n+1/2},x), v - \Delta t E(t^{n+1/2},x))\} \\ \\ \partial_v f(t^n, x - v\frac{\Delta t}{2} + \frac{\Delta t^2}{2}E(t^{n+1/2},x), v - \Delta t E(t^{n+1/2},x)) \\ -\frac{\Delta t}{2}\partial_x f(t^n, x - v\frac{\Delta t}{2} + \frac{\Delta t^2}{2}E(t^{n+1/2},x), v - \Delta t E(t^{n+1/2},x)) \end{pmatrix}$$
$$= \begin{pmatrix} f(t^n, \widetilde{X}(t^n), \widetilde{V}(t^n)) \\ \dfrac{\partial \widetilde{X}}{\partial x}(t^n)\partial_x f(t^n, \widetilde{X}(t^n), \widetilde{V}(t^n)) + \dfrac{\partial \widetilde{V}}{\partial x}(t^n)\partial_v f(t^n, \widetilde{X}(t^n), \widetilde{V}(t^n)) \\ \dfrac{\partial \widetilde{X}}{\partial v}(t^n)\partial_x f(t^n, \widetilde{X}(t^n), \widetilde{V}(t^n)) + \dfrac{\partial \widetilde{V}}{\partial v}(t^n)\partial_v f(t^n, \widetilde{X}(t^n), \widetilde{V}(t^n)) \end{pmatrix},$$

where

$$\widetilde{X}(t^n) = x - v\Delta t + \frac{\Delta t^2}{2}E(t^{n+1/2}, x - v\Delta t/2)$$

and

$$\widetilde{V}(t^n) = v - \Delta t E(t^{n+1/2}, x - v\Delta t/2).$$

A Taylor expansion gives

$$\begin{aligned} X(t^{n+1/2}) - (x - v\Delta t/2) &= X(t^{n+1/2}) - \left(X(t^{n+1}) - V(t^{n+1}) \frac{\Delta t}{2} \right) \\ &= X(t^{n+1/2}) - \left(X(t^{n+1}) - \frac{\Delta t}{2} \dot{X}(t^{n+1}) \right) \\ &= \mathcal{O}(\Delta t^2). \end{aligned} \tag{4.1}$$

By using (4.1) we get

$$\begin{aligned} V(t^n) - \widetilde{V}(t^n) &= V(t^n) - \left(V(t^{n+1}) - \Delta t E\left(t^{n+1/2}, X(t^{n+1}) - V(t^{n+1}) \frac{\Delta t}{2} \right) \right) \\ &= V(t^n) - \left(V(t^{n+1}) - \Delta t E\left(t^{n+1/2}, X(t^{n+1/2}) + \mathcal{O}(\Delta t^2) \right) \right) \\ &= V(t^n) - (V(t^{n+1}) - \Delta t E(t^{n+1/2}, X(t^{n+1/2})) + \mathcal{O}(\Delta t^3) \\ &= V(t^n) - V(t^{n+1}) + \Delta t \dot{V}(t^{n+1/2}) + \mathcal{O}(\Delta t^3). \end{aligned}$$

We deduce that

$$\begin{aligned} &\sup \left\{ \left| \partial^\alpha V(t^n; t^{n+1}, x, v) - \partial^\alpha \widetilde{V}(t^n; t^{n+1}, x, v) \right| \mid \forall (x, v) \in [0, L] \times [-R, R],\ |\alpha| \le 1 \right\} \\ &\quad \le C\Delta t^3. \end{aligned}$$

In the same way we have

$$\begin{aligned} &X(t^n) - \widetilde{X}(t^n) \\ &\quad = X(t^n) - \left(X(t^{n+1}) - \Delta t V(t^{n+1}) + \frac{\Delta t^2}{2} E\left(t^{n+1/2}, X(t^{n+1}) - V(t^{n+1}) \frac{\Delta t}{2} \right) \right) \\ &\quad = X(t^n) - \left(X(t^{n+1}) - \Delta t V(t^{n+1}) + \frac{\Delta t^2}{2} E(t^{n+1/2}, X(t^{n+1/2}) + \mathcal{O}(\Delta t^2)) \right) \\ &\quad = X(t^n) - \left(X(t^{n+1}) - \Delta t V(t^{n+1}) + \frac{\Delta t^2}{2} E(t^{n+1/2}, X(t^{n+1/2})) \right) + \mathcal{O}(\Delta t^4) \\ &\quad = X(t^n) - \left(X(t^{n+1}) - \Delta t \dot{X}(t^{n+1}) + \frac{\Delta t^2}{2} \ddot{X}(t^{n+1/2}) \right) + \mathcal{O}(\Delta t^4) \\ &\quad = X(t^n) - \left(X(t^{n+1}) - \Delta t \dot{X}(t^{n+1}) + \frac{\Delta t^2}{2} \ddot{X}(t^{n+1}) \right) + \mathcal{O}(\Delta t^3). \end{aligned}$$

It follows that

$$\sup \left\{ \left| \partial^\alpha X(t^n) - \partial^\alpha \widetilde{X}(t^n) \right| \mid \forall (x, v) \in [0, L] \times [-R, R],\ |\alpha| \le 1 \right\} \le C\Delta t^3.$$

We note that

$$X_{k,l}(t^n) = X(t^n; t^{n+1}, x_k, v_l), \qquad V_{k,l}(t^n) = V(t^n; t^{n+1}, x_k, v_l).$$

Finally we get

$$\begin{aligned}(\mathcal{T}_1 \circ \mathcal{T}_2 \circ \mathcal{T}_1 f(t^n))_{k,l} &= f(t^n, X_{k,l}(t^n) + \mathcal{O}(\Delta t^3), V_{k,l}(t^n) + \mathcal{O}(\Delta t^3)) \\ &= f(t^n, X_{k,l}(t^n), V_{k,l}(t^n)) + \nabla f(t^n, X_{k,l}(t^n), V_{k,l}(t^n)).\mathcal{O}(\Delta t^3) \\ &= f(t^{n+1}, X_{k,l}(t^{n+1}), V_{k,l}(t^{n+1})) + \nabla f(t^n, X_{k,l}(t^n), V_{k,l}(t^n)).\mathcal{O}(\Delta t^3) \\ &= f(t^{n+1}, x_k, v_l) + \nabla f(t^n, X_{k,l}(t^n), V_{k,l}(t^n)).\mathcal{O}(\Delta t^3)\end{aligned}$$

and

$$\|f(t^{n+1}) - \mathcal{T}_1 \circ \mathcal{T}_2 \circ \mathcal{T}_1 f(t^n)\|_{L^2_h(\Omega)} \leq C\Delta t^3 \|\nabla f\|_{l^\infty(0,T;L^\infty(\Omega))}.$$

Moreover we have

$$\begin{aligned}&\left|\nabla f(t^{n+1}, x_k, v_l) - (\nabla\mathcal{T}_1 \circ \nabla\mathcal{T}_2 \circ \nabla\mathcal{T}_1 f(t^n))_{k,l}\right| \\ &\quad \leq \left|\left(\nabla\widetilde{X}_{k,l}(t^n), \nabla\widetilde{V}_{k,l}(t^n)\right) - (\nabla X_{k,l}(t^n), \nabla V_{k,l}(t^n))\right| \\ &\qquad \cdot \left|\nabla f\left(t^n, \widetilde{X}_{k,l}(t^n), \widetilde{V}_{k,l}(t^n)\right)\right| + |(\nabla X_{k,l}(t^n), \nabla V_{k,l}(t^n))| \\ &\qquad \cdot \left|\nabla f\left(t^n, \widetilde{X}_{k,l}(t^n), \widetilde{V}_{k,l}(t^n)\right) - \nabla f\left(t^n, X_{k,l}(t^n), V_{k,l}(t^n)\right)\right|,\end{aligned}$$

so that we get

$$\begin{aligned}&\left\|\nabla f(t^{n+1}) - \nabla\mathcal{T}_1 \circ \nabla\mathcal{T}_2 \circ \nabla\mathcal{T}_1 f(t^n)\right\|_{\mathbb{L}^2_h(\Omega)} \\ &\quad \leq C\Delta t^3 \left(\|\nabla f\|_{l^\infty(0,T;L^\infty(\Omega))} + \left\|\nabla^2 f\right\|_{l^\infty(0,T;L^\infty(\Omega))}\right),\end{aligned}$$

which ends the proof. □

We go on with a proposition which gives information on the L^2_h-stability of the interpolation operators.

PROPOSITION 4.4. *Let $\mathcal{H}_h$ and $\mathcal{L}_h$ be the interpolation operators and $\tau_{\boldsymbol{\mu},\boldsymbol{\nu}}$ the translation operator defined in section 3.1. Then for all functions $f \in \mathscr{C}\left(0,T;\mathscr{C}^1_b(\Omega)\right)$, we get the estimates*

$$\|\tau_{\boldsymbol{\mu},0}\mathcal{L}_h f\|_{L^2_h(\Omega)} \leq \|f\|_{L^2_h(\Omega)}, \tag{4.2}$$

$$\|\tau_{\boldsymbol{\mu},0}\mathcal{H}_h f\|_{L^2_h(\Omega)} \leq \|f\|_{L^2_h(\Omega)}, \tag{4.3}$$

$$\|\tau_{\boldsymbol{\mu},0}\partial_x\mathcal{H}_h f\|_{L^2_h(\Omega)} \leq \|\partial_x f\|_{L^2_h(\Omega)}, \tag{4.4}$$

$$\|\tau_{0,\boldsymbol{\nu}}\mathcal{L}_h f\|_{L^2_h(\Omega)} \leq \|f\|_{L^2_h(\Omega)}, \tag{4.5}$$

$$\|\tau_{0,\boldsymbol{\nu}}\mathcal{H}_h f\|_{L^2_h(\Omega)} \leq \|f\|_{L^2_h(\Omega)}, \tag{4.6}$$

$$\|\tau_{0,\boldsymbol{\nu}}\partial_v\mathcal{H}_h f\|_{L^2_h(\Omega)} \leq \|\partial_v f\|_{L^2_h(\Omega)}. \tag{4.7}$$

Proof. We give only the proof for the estimates (4.2)–(4.4), because the proof of inequalities (4.5)–(4.7) is the same.

Let us start with the estimate (4.2). We have

$$\|\tau_{\boldsymbol{\mu},0}\mathcal{L}_h f\|_{L^2_h(\Omega)} = \|\mathcal{L}_h f\|_{L^2_h,\Delta^{\boldsymbol{\mu},0}_h}.$$

Let $\boldsymbol{\omega} = (\omega_x, \omega_v)$ be a two-component vector belonging to $\mathbb{Z}^2$. $\mathbf{z}$ is an abbreviation for (x, v), and $\mathbf{k}$ is an abbreviation for (k_x, k_v). k_x (resp., k_v) takes the values $2\pi\omega_x/L$

(resp., $2\pi\omega_v/(2R)$), where ω_x and ω_v are integers. If we note that $\mathbf{N} = (N_x, N_v)$ and $\mathbf{z}_{k,l} = (x_k, v_l)$, the Fourier series decomposition of f gives

$$f_{i,j} = \frac{1}{|\Omega|^{1/2}} \sum_{|\boldsymbol{\omega}| \le \mathbf{N}/2} \widehat{f}(\boldsymbol{\omega}) e^{i\langle \mathbf{k}(\boldsymbol{\omega}), \mathbf{z}_{i,j}\rangle}.$$

Therefore we get

$$\begin{aligned}(\mathcal{L}_h f)_{i+\mu_j,j} &= \frac{1}{|\Omega|^{1/2}} \sum_{|\boldsymbol{\omega}| \le \mathbf{N}/2} \sum_{k=i-1}^{i+2} \widehat{f}(\boldsymbol{\omega}) \ell_k^3(x_i + \mu_j \Delta x) e^{i\langle \mathbf{k}(\boldsymbol{\omega}), \mathbf{z}_{k,j}\rangle} \\ &= \frac{1}{|\Omega|^{1/2}} \sum_{|\boldsymbol{\omega}| \le \mathbf{N}/2} \widehat{f}(\boldsymbol{\omega}) \varrho(\mu_j, \omega_x) e^{i\langle \mathbf{k}(\boldsymbol{\omega}), \mathbf{z}_{i-1,j}\rangle},\end{aligned}$$

where

$$\varrho(\mu_j, \omega_x) = \sum_{k=0}^{3} \ell_k^3((1+\mu_j)\Delta x) e^{ik_x(\omega_x)x_k}.$$

As a consequence we obtain

$$\begin{aligned}\|\mathcal{L}_h f\|^2_{L^2_h, \Delta_h^{\boldsymbol{\mu},0}} &= \Delta x \Delta v \sum_{i=0}^{N_x} \sum_{j=0}^{N_v} (\mathcal{L}_h f)_{i+\mu_j,j} \overline{(\mathcal{L}_h f)}_{i+\mu_j,j} \\ &= \frac{\Delta x \Delta v}{|\Omega|} \sum_{i=0}^{N_x} \sum_{j=0}^{N_v} \sum_{|\boldsymbol{\omega}| \le \mathbf{N}/2} \sum_{|\boldsymbol{\omega}'| \le \mathbf{N}/2} \widehat{f}(\boldsymbol{\omega}) \overline{\widehat{f}(\boldsymbol{\omega}')} \varrho(\mu_j, \omega_x) \overline{\varrho(\mu_j, \omega_x')} \\ &\qquad e^{-i\langle \mathbf{k}(\boldsymbol{\omega}) - \mathbf{k}(\boldsymbol{\omega}'), (1,0)\rangle} e^{i\langle \mathbf{k}(\boldsymbol{\omega}) - \mathbf{k}(\boldsymbol{\omega}'), \mathbf{z}_{i,j}\rangle}.\end{aligned}$$

Since

$$\frac{\Delta x}{|L|} \sum_{i=0}^{N_x} e^{i(k_x(\omega_x) - k_x(\omega_x'))x_i} = \delta_{\omega_x, \omega_x'},$$

we find

$$\|\mathcal{L}_h f\|^2_{L^2_h, \Delta_h^{\boldsymbol{\mu},0}} = \frac{\Delta v}{|2R|} \sum_{j=0}^{N_v} \sum_{|\boldsymbol{\omega}| \le \mathbf{N}/2} \sum_{|\omega_v'| \le N_v/2} \widehat{f}(\boldsymbol{\omega}) \overline{\widehat{f}(\omega_x, \omega_v')} |\varrho(\mu_j, \omega_x)|^2 e^{i(k_v(\omega_v) - k_v(\omega_v'))v_j}.$$

Finally as

$$\frac{\Delta v}{|2R|} \sum_{j=0}^{N_v} e^{i(k_v(\omega_v) - k_v(\omega_v'))v_j} = \delta_{\omega_v, \omega_v'},$$

we get

$$\begin{aligned}\|\mathcal{L}_h f\|^2_{L^2_h, \Delta_h^{\boldsymbol{\mu},0}} &\le \sup\{|\varrho(\mu, \omega_x)|^2,\ |\omega_x \Delta x| \le L/2,\ 0 \le \mu \le 1\} \\ &\quad \times \frac{\Delta v}{|2R|} \sum_{j=0}^{N_v} \sum_{|\boldsymbol{\omega}| \le \mathbf{N}/2} \sum_{|\omega_v'| \le N_v/2} \widehat{f}(\boldsymbol{\omega}) \overline{\widehat{f}(\omega_x, \omega_v')} e^{i(k_v(\omega_v) - k_v(\omega_v'))v_j} \\ &\le \sup\{|\varrho(\mu, \omega_x)|^2,\ |\omega_x \Delta x| \le L/2,\ 0 \le \mu \le 1\} \sum_{|\boldsymbol{\omega}| \le \mathbf{N}/2} |\widehat{f}(\boldsymbol{\omega})|^2 \\ &\le \sup\{|\varrho(\mu, \omega_x)|^2,\ |\omega_x \Delta x| \le L/2,\ 0 \le \mu \le 1\} \|f\|^2_{L^2_h}.\end{aligned}$$

Without loss of generality we can suppose that $L = 2\pi$, $\Delta x = h$, $\xi = \omega_x h$, and $\zeta = \mu - 1/2$. Then we have to prove that

$$\sup\{|\varrho(\zeta,\xi)|,\ |\xi| \le \pi,\ |\zeta| \le 1/2\} \le 1,$$

with

$$\varrho(\zeta,\xi) = \sum_{k=-3/2}^{3/2} \ell_k^3(\zeta) e^{ik\xi},$$

and

$$\ell_k^3(\zeta) = \prod_{\substack{i=-3/2\\ i\neq k}}^{3/2} \frac{(\zeta - i)}{(k-i)},\ k \in \{-3/2, -1/2, 1/2, 3/2\}.$$

If we set $\theta = 1 - \cos(\xi)$, after some algebra we find

$$|\varrho(\zeta,\theta)|^2 = 1 - \left(\left(\frac{1}{2}\right)^2 - \zeta^2\right)\left(\left(\frac{3}{2}\right)^2 - \zeta^2\right)\theta^2\left[3 + 2\theta\left(\left(\frac{1}{2}\right)^2 - \zeta^2\right)\right]\Big/9,$$

which ends the proof of the estimate (4.2).

Proceeding in the same way, we get

$$\begin{aligned}\|\tau_{\boldsymbol{\mu},0}\mathcal{H}_h f\|_{L_h^2(\Omega)} &= \|\mathcal{H}_h f\|_{L_h^2,\Delta_h^{\boldsymbol{\mu},0}}\\ &\le \sup\{|\mathcal{Q}(\mu,\xi,h)|,\ 0\le\mu\le 1,\ |\xi|\le\pi\}\left(\sum_{|\boldsymbol{\omega}|\le \mathbf{N}/2} |\widehat{f}(\boldsymbol{\omega})|^2\right)^{1/2}\\ &\le \sup\{|\mathcal{Q}(\mu,\xi,h)|,\ 0\le\mu\le 1,\ |\xi|\le\pi\}\|f\|_{L_h^2},\end{aligned}$$

where

$$\mathcal{Q}(\mu,\xi,h) = \phi_0(\mu h) + \phi_1(\mu h)e^{i\xi} + i\frac{\xi}{h}\psi_0(\mu h) + i\frac{\xi}{h}\psi_1(\mu h)e^{i\xi}$$

and

$$\begin{aligned}\|\tau_{\boldsymbol{\mu},0}\partial_x\mathcal{H}_h f\|_{L_h^2(\Omega)} &= \|\partial_x\mathcal{H}_h f\|_{L_h^2,\Delta_h^{\boldsymbol{\mu},0}}\\ &\le \sup\{|\widehat{\mathcal{Q}}(\mu,\xi,h)|,\ 0\le\mu\le 1,\ |\xi|\le\pi\}\left(\sum_{|\boldsymbol{\omega}|\le \mathbf{N}/2} |\omega_x|^2|\widehat{f}(\boldsymbol{\omega})|^2\right)^{1/2}\\ &\le \sup\{|\widehat{\mathcal{Q}}(\mu,\xi,h)|,\ 0\le\mu\le 1,\ |\xi|\le\pi\}\|\partial_x f\|_{L_h^2},\end{aligned}$$

where

$$|\widehat{\mathcal{Q}}(\mu,\xi,h)|\cdot|\xi| = |\dot{\mathcal{Q}}(\mu,\xi,h)|,$$

with

$$\dot{\mathcal{Q}}(\mu,\xi,h) = \dot{\phi}_0(\mu h) + \dot{\phi}_1(\mu h)e^{i\xi} + i\xi\dot{\psi}_0(\mu h) + i\xi\dot{\psi}_1(\mu h)e^{i\xi}.$$

Therefore we have to prove that

$$\mathcal{Q}_{\text{sup}} = \sup\{|\mathcal{Q}(\mu, \xi, h)|,\ 0 \le \mu \le 1,\ |\xi| \le \pi\} \le 1$$

and

$$\widehat{\mathcal{Q}}_{\text{sup}} = \sup\{|\widehat{\mathcal{Q}}(\mu, \xi, h)|,\ 0 \le \mu \le 1,\ |\xi| \le \pi\} \le 1.$$

Let us start by $\mathcal{Q}_{\text{sup}}$. First we note that $\mathcal{Q}$ can be rewritten as

$$\mathcal{Q}(\mu, \xi, h) = \mathcal{Q}(\mu, \xi) = (1+2\mu)(1-\mu)^2 + (3-2\mu)\mu^2 e^{i\xi} + i\xi\mu(1-\mu)^2 + i\xi\mu^2(\mu-1)e^{i\xi}.$$

If we take the modulus of $\mathcal{Q}$, after some algebra we get

$$\begin{aligned}|\mathcal{Q}(\mu, \xi)|^2 = 1 - \mu^2(1-\mu)^2 \{&2(1-\cos\xi)(3-2\mu)(1+2\mu) - \xi^2(4\mu^2 - 4\mu + 1)\\ &- 2\,\xi^2(1-\cos\xi)\mu(1-\mu) - 2\xi\sin\xi(-4\mu^2 + 4\mu + 1)\}.\end{aligned}$$

If we make the change of variable $\nu = \mu - 1/2$, after some algebra we get

$$|\mathcal{Q}(\nu, \xi)|^2 = 1 - \left(\left(\frac{1}{2}\right)^2 - \nu^2\right)^2 \sigma(\nu, \xi),$$

where

$$\sigma(\nu, \xi) = -\nu^2(4\sin(\xi/2) - 2\xi\sin(\xi/2))^2 + 8(1-\cos\xi) - \frac{1}{2}\xi^2(1-\cos\xi) - 4\xi\sin^2\xi.$$

In order to show that $\mathcal{Q}_{\text{sup}} \le 1$ it is enough to show that $\sigma \ge 0$ for all $|\xi| \le \pi,\ |\nu| \le 1/2$. Since σ is a polynomial of degree two with respect to the variable ν whose term in ν^2 is negative, we have to prove that for all ξ such that $|\xi| \le \pi$ the roots of σ are real and of modulus greater than or equal to 1/2. The roots of σ are

$$\nu_r(\xi) = \pm\sqrt{\frac{-8(1-\cos\xi) + \frac{1}{2}\xi^2(1-\cos\xi) + 4\xi\sin^2\xi}{-\nu^2(4\sin(\xi/2) - 2\xi\sin(\xi/2))^2}}.$$

We must prove that $|\nu_r(\xi)|^2 > \frac{1}{4}$, which is equivalent to showing that $g(\xi) \le 0$, where $g(\xi) = -6(1-\cos\xi) + \xi^2 + 2\xi\sin\xi$. In order to prove that g is negative on $[-\pi, \pi]$, we study the function g. For that we need to derive g three times.

Now, let us show that $\widehat{Q}_{sup} \le 1$. In order to prove that $\widehat{Q}_{sup} \le 1$ it is enough to prove that

$$|\dot{\mathcal{Q}}(\mu, \xi, h)|^2 - |\xi|^2 \le 0 \quad \forall |\xi| \le \pi,\ 0 \le \mu \le 1.$$

In fact $\dot{\mathcal{Q}}(\mu, \xi, h)$ can be recast as

$$\begin{aligned}\dot{\mathcal{Q}}(\mu, \xi, h) &= \dot{\mathcal{Q}}(\mu, \xi)\\ &= 6\mu(\mu-1) + 6\mu(1-\mu)e^{i\xi} + i\xi(1-\mu)(1-3\mu) + i\xi\mu(3\mu-2)e^{i\xi}.\end{aligned}$$

Taking the modulus of $\dot{\mathcal{Q}}$, we get

$$\begin{aligned}|\dot{\mathcal{Q}}(\mu, \xi)|^2 = {}&72\mu^2(1-\mu)^2(1-\cos\xi) + 12\xi\sin\xi\mu(1-\mu)(6\mu^2 - 6\mu + 1)\\ &+ \xi^2(1-6\mu+6\mu^2)^2 + 2(\cos\xi - 1)\xi^2\mu(1-\mu)(1-3\mu)(3\mu-2).\end{aligned}$$

If we make the change of variable $\nu = \mu - 1/2$, after some algebra we find

$$|\dot{\mathcal{Q}}(\nu,\xi)|^2 - |\xi|^2 = a(\xi)\nu^4 + b(\xi)\nu^2 + c(\xi),$$

where

$$a(\xi) = \left[12\sin\frac{\xi}{2} - 6\xi\cos\frac{\xi}{2}\right]^2, \quad c(\xi) = \left[\left(3\sin\frac{\xi}{2} - \frac{\xi}{2}\cos\frac{\xi}{2}\right)^2 - \xi^2\right],$$

$$b(\xi) = \left[-2\left(6\sin\frac{\xi}{2} - 2\xi\cos\frac{\xi}{2}\right)^2 + \xi^2\left(1+\sin^2\frac{\xi}{2}\right)\right].$$

First we note that $|\dot{\mathcal{Q}}(\pm\frac{1}{2},\xi)|^2 - |\xi|^2 = 0$. Consequently $|\dot{\mathcal{Q}}|^2 - |\xi|^2$ can be rewritten as follows:

$$|\dot{\mathcal{Q}}(\nu,\xi)|^2 - |\xi|^2 = \left(\nu^2 - \left(\frac{1}{2}\right)^2\right)(\nu^2 + \beta(\xi)).$$

Let us show that β is positive. By following a classical result on the equation of second degree we get $\beta(\xi) = -4a(\xi)c(\xi)$. It remains to prove that $c(\xi) \le 0$ for all $|\xi| \le \pi$. We have $c(\xi) = l_1(\xi)l_2(\xi)$, where

$$l_1(\xi) = 3\sin\frac{\xi}{2} - \frac{\xi}{2}\cos\frac{\xi}{2} - \xi, \quad l_2(\xi) = 3\sin\frac{\xi}{2} - \frac{\xi}{2}\cos\frac{\xi}{2} + \xi.$$

In order to prove that $l_1 l_2$ is negative on $[-\pi,\pi]$, we study the sign of l_1 and l_2. For that we need to compute the derivatives of l_1 and l_2 until the order three. Finally we get

$$|\dot{\mathcal{Q}}(\nu,\xi)|^2 - |\xi|^2 \le 0, \quad |\nu| \le 1/2, \quad |\xi| \le \pi,$$

which ends the proof of the proposition. ☐

We continue with the following lemma, which gives an estimate of the term (I2).

LEMMA 4.5. *Let* $f \in \mathscr{C}_b\left(0,T;\mathscr{C}_b^2(\Omega)\right)$, *and then*

$$\left\|\mathcal{T}_1^{0,0}\circ\mathcal{T}_2^{0,0}\circ\mathcal{T}_1^{0,0}f(t^n) - \mathcal{T}_1^{0,0}\circ\widehat{\mathcal{T}}_2^{0,0}\circ\mathcal{T}_1^{0,0}f(t^n)\right\|_{L_h^2(\Omega)}$$
$$\le C\Delta t\left(\Delta t^2 + \Delta x^4 + \Delta v^4 + \Delta x^5 + \Delta x\Delta v^4 + \left\|\left(\mathcal{T}_1^{0,0} - \widetilde{\mathcal{T}}_1^{0,0}\right)f(t^n)\right\|_{L_h^2(\Omega)}\right.$$
$$+\left\|\widetilde{\mathcal{T}}_1^{0,0}\left(f(t^n) - f_h(t^n)\right)\right\|_{L_h^2(\Omega)} + \Delta x\left\|\left(\mathcal{T}_1^{1,0} - \widetilde{\mathcal{T}}_1^{1,0}\right)f(t^n)\right\|_{L_h^2(\Omega)}$$
$$\left.+\Delta x\left\|\widetilde{\mathcal{T}}_1^{1,0}\left(f(t^n) - f_h(t^n)\right)\right\|_{L_h^2(\Omega)}\right)$$

and

$$\left\|\nabla\mathcal{T}_1\circ\nabla\mathcal{T}_2\circ\nabla\mathcal{T}_1 f(t^n) - \nabla\mathcal{T}_1\circ\nabla\widehat{\mathcal{T}}_2\circ\nabla\mathcal{T}_1 f(t^n)\right\|_{\mathbb{L}_h^2(\Omega)}$$
$$\le C\Delta t\left(\Delta t^2 + \Delta x^3 + \Delta x^{-1}\Delta v^4 + \Delta v^4 + \Delta x^4 + \Delta x^{-1}\left\|\left(\mathcal{T}_1^{0,0} - \widetilde{\mathcal{T}}_1^{0,0}\right)f(t^n)\right\|_{L_h^2(\Omega)}\right.$$
$$+\Delta x^{-1}\left\|\widetilde{\mathcal{T}}_1^{0,0}\left(f(t^n) - f_h(t^n)\right)\right\|_{L_h^2(\Omega)} + \left\|\left(\mathcal{T}_1^{1,0} - \widetilde{\mathcal{T}}_1^{1,0}\right)f(t^n)\right\|_{L_h^2(\Omega)}$$
$$\left.+\left\|\widetilde{\mathcal{T}}_1^{1,0}\left(f(t^n) - f_h(t^n)\right)\right\|_{L_h^2(\Omega)}\right).$$

Proof. We introduce the characteristic curves $(\widehat{X}(t^n), \widehat{V}(t^n))$ defined by

$$\widehat{X}(t^n, x, v) = x - v\Delta t - \frac{1}{2}\Delta t^2 E_h^{n+1/2}(x - v\Delta t/2),$$
$$\widehat{V}(t^n, x, v) = v - \Delta t E_h^{n+1/2}(x - v\Delta t/2).$$

As has been done in Lemma 4.3 we have

$$\left(\mathcal{T}_1^{0,0} \circ \widehat{\mathcal{T}}_2^{0,0} \circ \mathcal{T}_1^{0,0} f(t^n)\right)_{k,l} = f\left(t^n, \widehat{X}_{k,l}(t^n), \widehat{V}_{k,l}(t^n)\right),$$
$$\left(\nabla\mathcal{T}_1 \circ \nabla\widehat{\mathcal{T}}_2 \circ \nabla\mathcal{T}_1 f(t^n)\right)_{k,l} = \nabla\left(\widehat{X}_{k,l}(t^n), \widehat{V}_{k,l}(t^n)\right)\nabla f\left(t^n, \widehat{X}_{k,l}(t^n), \widehat{V}_{k,l}(t^n)\right).$$

As a consequence we can write

(4.8)
$$\left|\left(\mathcal{T}_1^{0,0} \circ \mathcal{T}_2^{0,0} \circ \mathcal{T}_1^{0,0} f(t^n)\right)_{k,l} - \left(\mathcal{T}_1^{0,0} \circ \widehat{\mathcal{T}}_2^{0,0} \circ \mathcal{T}_1^{0,0} f(t^n)\right)_{k,l}\right|$$
$$\leq \left|\nabla f\left(\widetilde{X}_{k,l}(t^n), \widetilde{V}_{k,l}(t^n)\right)\right| \cdot \left(\left|\widehat{X}_{k,l}(t^n) - \widetilde{X}_{k,l}(t^n)\right| + \left|\widehat{V}_{k,l}(t^n) - \widetilde{V}_{k,l}(t^n)\right|\right)$$

and

(4.9)
$$\left|(\nabla\mathcal{T}_1 \circ \nabla\mathcal{T}_2 \circ \nabla\mathcal{T}_1 f(t^n))_{k,l} - \left(\nabla\mathcal{T}_1 \circ \nabla\widehat{\mathcal{T}}_2 \circ \nabla\mathcal{T}_1 f(t^n)\right)_{k,l}\right|$$
$$\leq \left|\nabla f\left(\widetilde{X}_{k,l}(t^n), \widetilde{V}_{k,l}(t^n)\right)\right| \cdot \left(\left|\nabla\widehat{X}_{k,l}(t^n) - \nabla\widetilde{X}_{k,l}(t^n)\right| + \left|\nabla\widehat{V}_{k,l}(t^n) - \nabla\widetilde{V}_{k,l}(t^n)\right|\right)$$
$$+ \left(\left|\widehat{X}_{k,l}(t^n) - \widetilde{X}_{k,l}(t^n)\right| + \left|\widehat{V}_{k,l}(t^n) - \widetilde{V}_{k,l}(t^n)\right|\right)$$
$$\cdot \left|\nabla^2 f\left(\widetilde{X}_{k,l}(t^n), \widetilde{V}_{k,l}(t^n)\right)\right| \cdot \left|\nabla\left(\widetilde{X}_{k,l}(t^n), \widetilde{V}_{k,l}(t^n)\right)\right|.$$

Let us prove a bound for $|\partial^\alpha \widehat{X}_{k,l}(t^n) - \partial^\alpha \widetilde{X}_{k,l}(t^n)|$ and $|\partial^\alpha \widehat{V}_{k,l}(t^n) - \partial^\alpha \widetilde{V}_{k,l}(t^n)|$. By using the definition of $(\widetilde{X}(t^n), \widetilde{V}(t^n))$ and of $(\widehat{X}(t^n), \widehat{V}(t^n))$ we get

(4.10)
$$\left(\left|\widehat{X}_{k,l}(t^n) - \widetilde{X}_{k,l}(t^n)\right| + \left|\widehat{V}_{k,l}(t^n) - \widetilde{V}_{k,l}(t^n)\right|\right)$$
$$\leq C\Delta t \left\|E_h^{n+1/2}(x) - E(t^{n+1/2}, x)\right\|_{L^\infty}$$
$$\leq C\Delta t \left(\left\|E(t^{n+1/2}, x) - \widetilde{E}(t^{n+1/2}, x)\right\|_{L^\infty} + \left\|\widetilde{E}(t^{n+1/2}, x) - E_h^{n+1/2}(x)\right\|_{L^\infty}\right),$$

$$\left(\left|\partial_x\widehat{X}_{k,l}(t^n) - \partial_x\widetilde{X}_{k,l}(t^n)\right| + \left|\partial_x\widehat{V}_{k,l}(t^n) - \partial_x\widetilde{V}_{k,l}(t^n)\right|\right) \tag{4.11}$$
$$\leq C\Delta t \left(\left\|\partial_x E(t^{n+1/2}, x) - \partial_x\widetilde{E}(t^{n+1/2}, x)\right\|_{L^\infty} + \left\|\partial_x\widetilde{E}(t^{n+1/2}, x) - \partial_x E_h^{n+1/2}(x)\right\|_{L^\infty}\right)$$

and

$$\left(\left|\partial_v\widehat{X}_{k,l}(t^n) - \partial_v\widetilde{X}_{k,l}(t^n)\right| + \left|\partial_v\widehat{V}_{k,l}(t^n) - \partial_v\widetilde{V}_{k,l}(t^n)\right|\right) \tag{4.12}$$
$$\leq C\Delta t^2 \left(\left\|\partial_x E(t^{n+1/2}, x) - \partial_x\widetilde{E}(t^{n+1/2}, x)\right\|_{L^\infty} + \left\|\partial_x\widetilde{E}(t^{n+1/2}, x) - \partial_x E_h^{n+1/2}(x)\right\|_{L^\infty}\right),$$

where $\widetilde{E}(t^{n+1/2},x)$ and $\partial_x\widetilde{E}(t^{n+1/2},x)$ are solutions of the Poisson problems

$$(\widetilde{\mathrm{P}})\quad \begin{cases} \left(\dfrac{d}{dx}\widetilde{E}\right)(t^{n+1/2},x) = \displaystyle\int_v \mathcal{T}_1 f(t^n,x,v)dv - 1 = \int_v f(t^n, x - v\Delta t/2, v)dv - 1, \\ \displaystyle\int_0^L \widetilde{E}(t^{n+1/2},x)dx = 0 \end{cases}$$

and

$$(\widetilde{\mathrm{P}}')\quad \begin{cases} \left(\dfrac{d}{dx}\left(\dfrac{d}{dx}\widetilde{E}\right)\right)(t^{n+1/2},x) = \displaystyle\int_v \mathcal{T}_1^{1,0} f(t^n,x,v)dv = \int_v \partial_x f(t^n, x - v\Delta t/2, v)dv, \\ \displaystyle\int_0^L \left(\dfrac{d}{dx}\widetilde{E}\right)(t^{n+1/2},x)dx = 0. \end{cases}$$

First we give an estimate for $\|E(t^{n+1/2},x)-\widetilde{E}(t^{n+1/2},x)\|_{L^\infty}$ and for $\|\partial_x E(t^{n+1/2},x)-\partial_x\widetilde{E}(t^{n+1/2},x)\|_{L^\infty}$. By using the Vlasov equation we get that

$$\begin{aligned} \frac{f(t^{n+1/2},x,v) - f(t^n, x - v\Delta t/2, v)}{\Delta t/2} &= \partial_t f(t^{n+1/2},x,v) + v\partial_x f(t^{n+1/2},x,v) + \mathcal{O}(\Delta t) \\ &= -E(t^{n+1/2},x)\partial_v f(t^{n+1/2},x,v) + \mathcal{O}(\Delta t), \end{aligned}$$

and it follows that

$$\begin{aligned} &\left\|E(t^{n+1/2}) - \widetilde{E}(t^{n+1/2})\right\|_{L^\infty} \\ &\qquad = \left\|\int_0^L K(x,y)\left(\int_{-\infty}^{+\infty}[f(t^{n+1/2},y,v) - f(t^n, y - v\Delta t/2, v)]dv\right)dy\right\|_{L^\infty} \\ &\qquad \le C\Delta t^2. \end{aligned} \tag{4.13}$$

In the same manner, if we derive the Vlasov equation with respect to the variable x, we get

$$\begin{aligned} \frac{\partial_x f(t^{n+1/2},x,v) - \partial_x f(t^n, x - v\Delta t/2, v)}{\Delta t/2} &= -\partial_x E(t^{n+1/2},x)\partial_v f(t^{n+1/2},x,v) \\ &\quad - E(t^{n+1/2},x)\partial_{vx} f(t^{n+1/2},x,v) + \mathcal{O}(\Delta t), \end{aligned}$$

and we find

$$\begin{aligned} &\left\|\partial_x E(t^{n+1/2}) - \partial_x\widetilde{E}(t^{n+1/2})\right\|_{L^\infty} \\ &\qquad = \left\|\int_0^L K(x,y)\left(\int_{-\infty}^{+\infty}[\partial_x f(t^{n+1/2},y,v) - \partial_x f(t^n, y - v\Delta t/2, v)]dv\right)dy\right\|_{L^\infty} \\ &\qquad \le C\Delta t^2. \end{aligned} \tag{4.14}$$

Now we estimate the terms $\|E_h^{n+1/2}(x) - \widetilde{E}(t^{n+1/2},x)\|_{L^\infty}$ and $\|\partial_x E_h^{n+1/2}(x) - \partial_x\widetilde{E}(t^{n+1/2},x)\|_{L^\infty}$. Until the end of the proof the following notation is used:

$$\left\|\widetilde{E}(t^{n+1/2}) - E_h^{n+1/2}\right\|_\infty = \max_k \left|\widetilde{E}(t^{n+1/2},x_k) - E_h^{n+1/2}(x_k)\right|,$$

$$\left|\widetilde{E}(t^{n+1/2}) - E_h^{n+1/2}\right|_{1,\infty} = \max_k \left|\partial_x \widetilde{E}(t^{n+1/2}, x_k) - \partial_x E_h^{n+1/2}(x_k)\right|,$$

and

$$\|\cdot\|_{1,\infty} = \|\cdot\|_{\infty} + |\cdot|_{1,\infty}.$$

Suppose that $x = x_k + \theta\Delta x$, with $0 \le \theta \le 1$, and then we get

$$\begin{aligned}
&\left|\widetilde{E}(t^{n+1/2}, x) - E_h^{n+1/2}(x)\right| \le \Big|\widetilde{E}(t^{n+1/2}, x) - \phi_0(\theta\Delta x)\widetilde{E}(t^{n+1/2}, x_k) \\
&- \phi_1(\theta\Delta x)\widetilde{E}(t^{n+1/2}, x_{k+1}) - \psi_0(\theta\Delta x)\partial_x\widetilde{E}(t^{n+1/2}, x_k) - \psi_1(\theta\Delta x)\partial_x\widetilde{E}(t^{n+1/2}, x_{k+1})\Big| \\
&+ \Big|\phi_0(\theta\Delta x)\left(\widetilde{E}(t^{n+1/2}, x_k) - E_h^{n+1/2}(x_k)\right) \\
&+ \phi_1(\theta\Delta x)\left(\widetilde{E}(t^{n+1/2}, x_{k+1}) - E_h^{n+1/2}(x_{k+1})\right) \\
&+\psi_0(\theta\Delta x)\left(\partial_x\widetilde{E}(t^{n+1/2}, x_k) - \partial_x E_h^{n+1/2}(x_k)\right) \\
&+ \psi_1(\theta\Delta x)\left(\partial_x\widetilde{E}(t^{n+1/2}, x_{k+1}) - \partial_x E_h^{n+1/2}(x_{k+1})\right)\Big|.
\end{aligned}$$

We deduce that

$$\begin{aligned}
\left\|\widetilde{E}(t^{n+1/2}) - E_h^{n+1/2}\right\|_{L^\infty} &\le C\Bigg(\Delta x^4 \left\|\widetilde{E}\right\|_{\mathscr{C}(0,T;\mathscr{C}^4([0,L]))} + \left\|\widetilde{E}(t^{n+1/2}) - E_h^{n+1/2}\right\|_{\infty} \\
&\quad + \Delta x \left|\widetilde{E}(t^{n+1/2}) - E_h^{n+1/2}\right|_{1,\infty}\Bigg).
\end{aligned} \tag{4.15}$$

In the same way we show that

$$\begin{aligned}
\left\|\partial_x\widetilde{E}(t^{n+1/2}) - \partial_x E_h^{n+1/2}\right\|_{L^\infty} &\le C\Bigg(\Delta x^4 \left\|\widetilde{E}\right\|_{\mathscr{C}(0,T;\mathscr{C}^4([0,L]))} \\
&+ \Delta x^{-1}\left\|\widetilde{E}(t^{n+1/2}) - E_h^{n+1/2}\right\|_{\infty} + \left|\widetilde{E}(t^{n+1/2}) - E_h^{n+1/2}\right|_{1,\infty}\Bigg).
\end{aligned} \tag{4.16}$$

Now we estimate the terms $\|\widetilde{E}(t^{n+1/2}) - E_h^{n+1/2}\|_\infty$ and $|\widetilde{E}(t^{n+1/2}) - E_h^{n+1/2}|_{1,\infty}$. We proceed as follows:

$$\begin{aligned}
&\frac{1}{12\Delta x}\left\{8\left(\widetilde{E}(t^{n+1/2}, x_{k+1}) - \widetilde{E}(t^{n+1/2}, x_{k-1})\right) - \left(\widetilde{E}(t^{n+1/2}, x_{k+2}) - \widetilde{E}(t^{n+1/2}, x_{k-2})\right)\right\} \\
&\qquad = \partial_x\widetilde{E}(t^{n+1/2}, x_k) + \mathcal{O}\left(\Delta x^4\right) \\
&\qquad = -1 + \int_v \mathcal{T}_1^{0,0} f(t^n, x_k, v)dv + \mathcal{O}\left(\Delta x^4\right) \\
&\qquad = -1 + \mathcal{O}\left(\Delta x^4\right) + \mathcal{O}\left(\Delta v^4\right) \\
&\qquad + \frac{\Delta v}{3}\Bigg\{\left(\mathcal{T}_1^{0,0} f(t^n)\right)_{k,0} + \left(\mathcal{T}_1^{0,0} f(t^n)\right)_{k,N_v} + 4\sum_{l,\text{odd}}\left(\mathcal{T}_1^{0,0} f(t^n)\right)_{k,l} \\
&\qquad\qquad + 2\sum_{l,\text{even}}\left(\mathcal{T}_1^{0,0} f(t^n)\right)_{k,l}\Bigg\}.
\end{aligned}$$

If we set $\delta E_k = \widetilde{E}(t^{n+1/2}, x_k) - E_h^{n+1/2}(x_k)$, we get

(4.17)
$$\begin{aligned}
&\frac{1}{12\Delta x}\left\{8\left(\delta E_{k+1}-\delta E_{k-1}\right)-\left(\delta E_{k+2}-\delta E_{k-2}\right)\right\}=\mathcal{O}\left(\Delta x^4\right)+\mathcal{O}\left(\Delta v^4\right)\\
&\quad+\frac{\Delta v}{3}\Bigg\{\left(\mathcal{T}_1^{0,0}f(t^n)\right)_{k,0}-\left(\widetilde{\mathcal{T}}_1^{0,0}f_h(t^n)\right)_{k,0}+\left(\mathcal{T}_1^{0,0}f(t^n)\right)_{k,N_v}-\left(\widetilde{\mathcal{T}}_1^{0,0}f_h(t^n)\right)_{k,N_v}\\
&\qquad+4\sum_{l,\text{odd}}\left[\left(\mathcal{T}_1^{0,0}f(t^n)\right)_{k,l}-\left(\widetilde{\mathcal{T}}_1^{0,0}f_h(t^n)\right)_{k,l}\right]\\
&\qquad+2\sum_{l,\text{even}}\left[\left(\mathcal{T}_1^{0,0}f(t^n)\right)_{k,l}-\left(\widetilde{\mathcal{T}}_1^{0,0}f_h(t^n)\right)_{k,l}\right]\Bigg\}.
\end{aligned}$$

By using the periodicity of δE_k we obtain

(4.18)
$$\begin{aligned}
&\frac{1}{12\Delta x}\sum_{k=0}^{N_x}\left|8(\delta E_{k+1}-\delta E_{k-1})-(\delta E_{k+2}-\delta E_{k-2})\right|\\
&\quad\geq\frac{1}{12\Delta x}\sum_{k=0}^{N_x}\left|8\left|\delta E_{k+1}-\delta E_{k-1}\right|-\left|\delta E_{k+2}-\delta E_{k-2}\right|\right|\\
&\quad\geq\frac{1}{12\Delta x}\left|8\sum_{k=0}^{N_x}\left|\delta E_{k+1}-\delta E_{k-1}\right|-\sum_{k=0}^{N_x}\left|\delta E_{k+2}-\delta E_{k-2}\right|\right|\\
&\quad\geq\frac{1}{12\Delta x}\left\{8\sum_{k=0}^{N_x}\left|\delta E_{k+1}-\delta E_{k-1}\right|-\sum_{k=0}^{N_x}\left|\delta E_{k+2}-\delta E_{k}\right|-\sum_{k=0}^{N_x}\left|\delta E_{k}-\delta E_{k-2}\right|\right\}\\
&\quad\geq\frac{1}{2\Delta x}\sum_{k=0}^{N_x}\left|\delta E_{k+1}-\delta E_{k-1}\right|.
\end{aligned}$$

By using (4.17), (4.18), and the Cauchy–Schwarz inequality we obtain

$$\begin{aligned}
(4.19)\quad&\sum_{k=0}^{N_x}\left|\delta E_{k+1}-\delta E_{k-1}\right|\leq\mathcal{O}\left(\Delta x^4\right)+\mathcal{O}\left(\Delta v^4\right)\\
&\quad+C\Delta x\Delta v\sum_{(k,l)\in\mathcal{M}_h}\left|\left(\mathcal{T}_1^{0,0}f(t^n)\right)_{k,l}-\left(\widetilde{\mathcal{T}}_1^{0,0}f_h(t^n)\right)_{k,l}\right|\\
&\quad\leq\mathcal{O}\left(\Delta x^4\right)+\mathcal{O}\left(\Delta v^4\right)+C\left(\left\|\left(\mathcal{T}_1^{0,0}-\widetilde{\mathcal{T}}_1^{0,0}\right)f(t^n)\right\|_{L_h^1(\Omega)}\right.\\
&\qquad\left.+\left\|\widetilde{\mathcal{T}}_1^{0,0}\left(f(t^n)-f_h(t^n)\right)\right\|_{L_h^1(\Omega)}\right)\\
&\quad\leq\mathcal{O}\left(\Delta x^4\right)+\mathcal{O}\left(\Delta v^4\right)+C\sqrt{|\Omega|}\left(\left\|\left(\mathcal{T}_1^{0,0}-\widetilde{\mathcal{T}}_1^{0,0}\right)f(t^n)\right\|_{L_h^2(\Omega)}\right.\\
&\qquad\left.+\left\|\widetilde{\mathcal{T}}_1^{0,0}\left(f(t^n)-f_h(t^n)\right)\right\|_{L_h^2(\Omega)}\right).
\end{aligned}$$

We set $\mathcal{M}$ as a mean value of δE_k

$$\mathcal{M} = \frac{\Delta v}{3L}\left\{\delta E_0 + \delta E_{N_x} + 4\sum_{l,\text{odd}}\delta E_l + 2\sum_{l,\text{even}}\delta E_l\right\},$$

and then

$$\min_k \ \delta E_k \le |\mathcal{M}|.$$

By using a discrete version of the Taylor formula with the integral remainder we get

$$\begin{aligned}\max_k \ \delta E_k &\le \min_k \ \delta E_k + \sum_{k=0}^{N_x}|\delta E_{k+1} - \delta E_{k-1}| \\ &\le |\mathcal{M}| + \sum_{k=0}^{N_x}|\delta E_{k+1} - \delta E_{k-1}|\,. \end{aligned} \tag{4.20}$$

Thanks to the discrete zero-mean electrostatic condition on E_h (see Problem (P_h)) we get

$$\begin{aligned}&\delta E_0 + \delta E_{N_x} + 4\sum_{l,\text{odd}}\delta E_l + 2\sum_{l,\text{even}}\delta E_l \\ &\quad = \widetilde{E}(t^{n+1/2},x_0) + \widetilde{E}(t^{n+1/2},x_{N_x}) + 4\sum_{l,\text{odd}}\widetilde{E}(t^{n+1/2},x_l) + 2\sum_{l,\text{even}}\widetilde{E}(t^{n+1/2},x_l) \\ &\quad -\left\{E_h^{n+1/2}(x_0) + E_h^{n+1/2}(x_{N_x}) + 4\sum_{l,\text{odd}}E_h^{n+1/2}(x_l) + 2\sum_{l,\text{even}}E_h^{n+1/2}(x_l)\right\} \\ &\quad = \int_0^L \widetilde{E}(t^{n+1/2},x)dx + \mathcal{O}(\Delta x^4) = \mathcal{O}(\Delta x^4). \end{aligned} \tag{4.21}$$

Finally, using (4.21) and (4.20) gives

$$|\delta E_k| \le \mathcal{O}(\Delta x^4) + \sum_{k=0}^{N_x}|\delta E_{k+1} - \delta E_{k-1}|\,. \tag{4.22}$$

Thanks to (4.22) and (4.19) we obtain

$$\begin{aligned}\left\|\widetilde{E}(t^{n+1/2}) - E_h^{n+1/2}\right\|_\infty &\le C\left(\Delta x^4 + \Delta v^4 + \left\|\left(\mathcal{T}_1^{0,0} - \widetilde{\mathcal{T}}_1^{0,0}\right)f(t^n)\right\|_{L_h^2(\Omega)}\right. \\ &\quad \left. + \left\|\widetilde{\mathcal{T}}_1^{0,0}\left(f(t^n) - f_h(t^n)\right)\right\|_{L_h^2(\Omega)}\right). \end{aligned} \tag{4.23}$$

In the same manner we get

$$\begin{aligned}\left|\widetilde{E}(t^{n+1/2}) - E_h^{n+1/2}\right|_{1,\infty} &\le C\left(\Delta x^4 + \Delta v^4 + \left\|\left(\mathcal{T}_1^{1,0} - \widetilde{\mathcal{T}}_1^{1,0}\right)f(t^n)\right\|_{L_h^2(\Omega)}\right. \\ &\quad \left. + \left\|\widetilde{\mathcal{T}}_1^{1,0}\left(f(t^n) - f_h(t^n)\right)\right\|_{L_h^2(\Omega)}\right). \end{aligned} \tag{4.24}$$

By putting together the estimates (4.8), (4.10), (4.13), (4.15), and (4.23)–(4.24) we obtain the first estimate of Lemma 4.5. Therefore by assembling the estimates (4.9)–(4.16) and (4.23)–(4.24) we get the second estimate of Lemma 4.5. ◻

We continue the proof, with the following proposition which states the H^1-stability of the transport operators.

PROPOSITION 4.6. *Let $f \in \mathscr{C}_b(0,T,\mathscr{C}_b^1(\Omega))$, and then there exists a constant C independent of Δx, Δv, and Δt such that*

$$\left\| \widetilde{\mathcal{T}}_1^{0,0} f(t) \right\|_{L^2_h(\Omega)} \leq \|f(t)\|_{L^2_h(\Omega)}, \tag{4.25}$$

$$\left\| \widetilde{\mathcal{T}}_1^{1,0} f(t) \right\|_{L^2_h(\Omega)} \leq \|\partial_x f(t)\|_{L^2_h(\Omega)}, \tag{4.26}$$

$$\left\| \widetilde{\mathcal{T}}_1^{0,1} f(t) \right\|_{L^2_h(\Omega)} \leq \|\partial_v f(t)\|_{L^2_h(\Omega)} + C\Delta t \|\partial_x f(t)\|_{L^2_h(\Omega)}, \tag{4.27}$$

$$\left\| \widetilde{\mathcal{T}}_2^{0,0} f(t) \right\|_{L^2_h(\Omega)} \leq \|f(t)\|_{L^2_h(\Omega)}, \tag{4.28}$$

$$\left\| \widetilde{\mathcal{T}}_2^{1,0} f(t) \right\|_{L^2_h(\Omega)} \leq \|\partial_x f(t)\|_{L^2_h(\Omega)} + C\Delta t \|\partial_v f(t)\|_{L^2_h(\Omega)}, \tag{4.29}$$

$$\left\| \widetilde{\mathcal{T}}_2^{0,1} f(t) \right\|_{L^2_h(\Omega)} \leq \|\partial_v f(t)\|_{L^2_h(\Omega)}. \tag{4.30}$$

Proof. By using (4.3) we get

$$\left\| \widetilde{\mathcal{T}}_1^{0,0} f \right\|_{L^2_h(\Omega)} = \left\| \mathbb{S}_{v\Delta t/2,0} \mathcal{H}_h f \right\|_{L^2_h(\Omega)} \leq \|f\|_{L^2_h(\Omega)}.$$

From (4.4) it follows that

$$\left\| \widetilde{\mathcal{T}}_1^{1,0} f \right\|_{L^2_h(\Omega)} = \left\| \mathbb{S}_{v\Delta t/2,0} \partial_x \mathcal{H}_h f \right\|_{L^2_h(\Omega)} \leq \|\partial_x f\|_{L^2_h(\Omega)}.$$

Thanks to (4.2) we find that

$$\begin{aligned}\left\| \widetilde{\mathcal{T}}_1^{0,1} f \right\|_{L^2_h(\Omega)} &\leq \left\| \mathbb{S}_{v\Delta t/2,0} \mathcal{L}_h \partial_v f \right\|_{L^2_h(\Omega)} + \frac{\Delta t}{2} \left\| \widetilde{\mathcal{T}}_1^{1,0} f \right\|_{L^2_h(\Omega)} \\ &\leq \|\partial_v f\|_{L^2_h(\Omega)} + C\Delta t \|\partial_x f\|_{L^2_h(\Omega)}.\end{aligned}$$

By using (4.3) we get

$$\left\| \widetilde{\mathcal{T}}_2^{0,0} f \right\|_{L^2_h(\Omega)} = \|\mathbb{S}_{0,\Delta t E_h} \mathcal{H}_h f\|_{L^2_h(\Omega)} \leq \|f\|_{L^2_h(\Omega)},$$

and from (4.6) it follows that

$$\left\| \widetilde{\mathcal{T}}_2^{0,1} f \right\|_{L^2_h(\Omega)} = \|\mathbb{S}_{0,\Delta t E_h} \partial_v \mathcal{H}_h f\|_{L^2_h(\Omega)} \leq \|\partial_v f\|_{L^2_h(\Omega)}.$$

Now let us show that $|E_h^{n+1/2}|_{1,\infty}$ is bounded. To do this, we suppose that $\|\partial_x f_h(t^n)\|_{L^2_h(\Omega)}$

is bounded. Then from Problem (P'_h) we get that

$$\begin{aligned}\sum_{k=0}^{N_x}\left|\partial_x E_h^{n+1/2}(x_{k+1})-\partial_x E_h^{n+1/2}(x_{k-1})\right| &\le 2L + C\Delta x\Delta v\sum_{k,l}\left|\left(\widetilde{\mathcal{T}}_1^{1,0}f_h(t^n)\right)_{k,l}\right|\\ &\le L + C\left\|\widetilde{\mathcal{T}}_1^{1,0}f_h(t^n)\right\|_{L_h^1(\Omega)}\\ &\le L + C\sqrt{|\Omega|}\left\|\widetilde{\mathcal{T}}_1^{1,0}f_h(t^n)\right\|_{L_h^2(\Omega)}\\ &\le C,\end{aligned}$$

and it follows that

$$\begin{aligned}\left|E_h^{n+1/2}\right|_{1,\infty} \le \frac{\Delta v}{3L}&\left|\partial_x E_h^{n+1/2}(x_0)+\partial_x E_h^{n+1/2}(x_{N_x})\right.\\ &\left.+\;4\sum_{l,\mathrm{odd}}\partial_x E_h^{n+1/2}(x_l)+2\sum_{l,\mathrm{even}}\partial_x E_h^{n+1/2}(x_l)\right|\\ &+\sum_{k=0}^{N_x}\left|\partial_x E_h^{n+1/2}(x_{k+1})-\partial_x E_h^{n+1/2}(x_{k-1})\right|\\ &\le C.\end{aligned}$$

Now let us prove inequality (4.29). By using (4.5) it follows that

$$\begin{aligned}\left\|\widetilde{\mathcal{T}}_2^{1,0}f\right\|_{L_h^2(\Omega)} &\le \left\|\mathbb{S}_{0,\Delta t E_h}\mathcal{L}_h\partial_x f\right\|_{L_h^2(\Omega)}+\frac{\Delta t}{2}\left\|\partial_x E_h^{n+1/2}\right\|_\infty\left\|\widetilde{\mathcal{T}}_2^{0,1}f\right\|_{L_h^2(\Omega)}\\ &\le \left\|\partial_x f\right\|_{L_h^2(\Omega)}+C\Delta t\left\|\partial_v f\right\|_{L_h^2(\Omega)},\end{aligned}$$

which ends the proof of the lemma. ∎

We continue with the following lemma, which gives an estimate of the precision of the approximation of the transport operators.

LEMMA 4.7. *Let $f\in\mathscr{C}_b(0,T;\mathscr{C}_b^4(\Omega))$, and then there exists a constant C independent of Δx, Δv, and Δt such that for $|\alpha|\le 1$ we have*

$$\begin{aligned}\left\|\left(\mathcal{T}_1^\alpha-\widetilde{\mathcal{T}}_1^\alpha\right)f(t)\right\|_{L_h^2(\Omega)} &\le C\Delta x^{4-|\alpha|}\left\|f\right\|_{\mathscr{C}_b\left(0,T;\mathscr{C}_b^4(\Omega)\right)},\\ \left\|\left(\widehat{\mathcal{T}}_2^\alpha-\widetilde{\mathcal{T}}_2^\alpha\right)f(t)\right\|_{L_h^2(\Omega)} &\le C\Delta v^{4-|\alpha|}\left\|f\right\|_{\mathscr{C}_b\left(0,T;\mathscr{C}_b^4(\Omega)\right)}.\end{aligned}$$

Proof. If $f\in\mathscr{C}^{2m}([a,b])$ and $\pi_h f$ is the Hermite polynomial of degree $2m-1$ which interpolates f and its $m-1$ first derivatives at the points a and b, then

$$\left\|f^{(j)}-\pi_h f^{(j)}\right\|_{L^\infty([a,b])}\le\frac{\left\|f^{(2m)}\right\|_{L^\infty([a,b])}}{(2m-j)!2^{2m-[j]}}h^{2m-j},\ 0\le j\le 2m-1,$$

where $[j]=j$ if j is even and $[j]=j+1$ if j is odd. Then we have

$$\begin{aligned}\left\|\left(\mathcal{T}_1^{0,0}-\widetilde{\mathcal{T}}_1^{0,0}\right)f(t)\right\|_{L_h^2(\Omega)} &\le C\left\|\mathbb{S}_{v\Delta t/2,0}\left(f(t)-\mathcal{H}_h f(t)\right)\right\|_{L_h^\infty(\Omega)}\\ &\le C\left\|\mathbb{S}_{v\Delta t/2,0}\left(f(t)-\mathcal{H}_h f(t)\right)\right\|_{L_x^\infty(L_{h,v}^\infty)}\\ &\le C\left\|f(t)-\mathcal{H}_h f(t)\right\|_{L_x^\infty(L_{h,v}^\infty)}\le C\Delta x^4\left\|f\right\|_{\mathscr{C}_b\left(0,T;\mathscr{C}_b^4(\Omega)\right)},\end{aligned}$$

$$\begin{aligned}\left\|\left(\mathcal{T}_1^{1,0}-\widetilde{\mathcal{T}}_1^{1,0}\right)f(t)\right\|_{L_h^2(\Omega)} &\le C\left\|\mathbb{S}_{v\Delta t/2,0}\left(\partial_x f(t)-\partial_x\mathcal{H}_h f(t)\right)\right\|_{L_h^\infty(\Omega)}\\ &\le C\left\|\mathbb{S}_{v\Delta t/2,0}\left(\partial_x f(t)-\partial_x\mathcal{H}_h f(t)\right)\right\|_{L_x^\infty(L_{h,v}^\infty)}\\ &\le C\left\|\partial_x f(t)-\partial_x\mathcal{H}_h f(t)\right\|_{L_x^\infty(L_{h,v}^\infty)}\le C\Delta x^3\|f\|_{\mathscr{C}_b\left(0,T;\mathscr{C}_b^4(\Omega)\right)},\end{aligned}$$

$$\begin{aligned}\left\|\left(\mathcal{T}_1^{0,1}-\widetilde{\mathcal{T}}_1^{0,1}\right)f(t)\right\|_{L_h^2(\Omega)} &\le C\left\|\mathbb{S}_{v\Delta t/2,0}\left(\partial_v f(t)-\mathcal{L}_h\partial_v f(t)\right)\right\|_{L_h^\infty(\Omega)}\\ &\quad + C\Delta t\left\|\mathbb{S}_{v\Delta t/2,0}\left(\partial_x f(t)-\partial_x\mathcal{H}_h f(t)\right)\right\|_{L_h^\infty(\Omega)}\\ &\le C\left\|\mathbb{S}_{v\Delta t/2,0}\left(\partial_v f(t)-\mathcal{L}_h\partial_v f(t)\right)\right\|_{L_x^\infty(L_{h,v}^\infty)}\\ &\quad + C\Delta t\left\|\mathbb{S}_{v\Delta t/2,0}\left(\partial_x f(t)-\partial_x\mathcal{H}_h f(t)\right)\right\|_{L_x^\infty(L_{h,v}^\infty)}\\ &\le C\left\|\partial_v f(t)-\mathcal{L}_h\partial_v f(t)\right\|_{L_x^\infty(L_{h,v}^\infty)}\\ &\quad + C\Delta t\left\|\partial_x f(t)-\partial_x\mathcal{H}_h f(t)\right\|_{L_x^\infty(L_{h,v}^\infty)}\\ &\le C(\Delta t\Delta x^3+\Delta x^4)\|f\|_{\mathscr{C}_b\left(0,T;\mathscr{C}_b^4(\Omega)\right)}\\ &\le C\Delta x^3\|f\|_{\mathscr{C}_b\left(0,T;\mathscr{C}_b^4(\Omega)\right)},\end{aligned}$$

$$\begin{aligned}\left\|\left(\widehat{\mathcal{T}}_2^{0,0}-\widetilde{\mathcal{T}}_2^{0,0}\right)f(t)\right\|_{L_h^2(\Omega)} &\le C\left\|\mathbb{S}_{0,\Delta tE_h}\left(f(t)-\mathcal{H}_h f(t)\right)\right\|_{L_h^\infty(\Omega)}\\ &\le C\left\|\mathbb{S}_{0,\Delta tE_h}\left(f(t)-\mathcal{H}_h f(t)\right)\right\|_{L_{h,x}^\infty(L_v^\infty)}\\ &\le C\left\|f(t)-\mathcal{H}_h f(t)\right\|_{L_{h,x}^\infty(L_v^\infty)}\le C\Delta v^4\|f\|_{\mathscr{C}_b\left(0,T;\mathscr{C}_b^4(\Omega)\right)},\end{aligned}$$

$$\begin{aligned}\left\|\left(\widehat{\mathcal{T}}_2^{0,1}-\widetilde{\mathcal{T}}_2^{0,1}\right)f(t)\right\|_{L_h^2(\Omega)} &\le C\left\|\mathbb{S}_{0,\Delta tE_h}\left(\partial_x f(t)-\partial_x\mathcal{H}_h f(t)\right)\right\|_{L_h^\infty(\Omega)}\\ &\le C\left\|\mathbb{S}_{0,\Delta tE_h}\left(\partial_x f(t)-\partial_x\mathcal{H}_h f(t)\right)\right\|_{L_{h,x}^\infty(L_v^\infty)}\\ &\le C\left\|\partial_x f(t)-\partial_x\mathcal{H}_h f(t)\right\|_{L_{h,x}^\infty(L_v^\infty)}\le C\Delta v^3\|f\|_{\mathscr{C}_b\left(0,T;\mathscr{C}_b^4(\Omega)\right)},\end{aligned}$$

$$\begin{aligned}\left\|\left(\widehat{\mathcal{T}}_2^{1,0}-\widetilde{\mathcal{T}}_2^{1,0}\right)f(t)\right\|_{L_h^2(\Omega)} &\le C\left\|\mathbb{S}_{0,\Delta tE_h}\left(\partial_x f(t)-\mathcal{L}_h\partial_x f(t)\right)\right\|_{L_h^\infty(\Omega)}\\ &\quad + C\Delta t\left\|\partial_x E_h^{n+1/2}\right\|_\infty\left\|\mathbb{S}_{0,\Delta tE_h}\left(\partial_v f(t)-\partial_v\mathcal{H}_h f(t)\right)\right\|_{L_h^\infty(\Omega)}\\ &\le C\left\|\mathbb{S}_{0,\Delta tE_h}\left(\partial_x f(t)-\mathcal{L}_h\partial_x f(t)\right)\right\|_{L_{h,x}^\infty(L_v^\infty)}\\ &\quad + C\Delta t\left\|\mathbb{S}_{0,\Delta tE_h}\left(\partial_v f(t)-\partial_v\mathcal{H}_h f(t)\right)\right\|_{L_{h,x}^\infty(L_v^\infty)}\\ &\le C\left\|\partial_x f(t)-\mathcal{L}_h\partial_x f(t)\right\|_{L_{h,x}^\infty(L_v^\infty)}\\ &\quad + C\Delta t\left\|\partial_v f(t)-\partial_v\mathcal{H}_h f(t)\right\|_{L_{h,x}^\infty(L_v^\infty)}\\ &\le C(\Delta t\Delta v^3+\Delta v^4)\|f\|_{\mathscr{C}_b\left(0,T;\mathscr{C}_b^4(\Omega)\right)}\\ &\le C\Delta v^3\|f\|_{\mathscr{C}_b\left(0,T;\mathscr{C}_b^4(\Omega)\right)}. \qquad \square\end{aligned}$$

Now let us evaluate the term (I3):

$$\left\|\mathcal{T}_1^\alpha\circ\widehat{\mathcal{T}}_2^\alpha\circ\mathcal{T}_1^\alpha f(t^n)-\widetilde{\mathcal{T}}_1^\alpha\circ\widetilde{\mathcal{T}}_2^\alpha\circ\widetilde{\mathcal{T}}_1^\alpha f(t^n)\right\|_{L_h^2(\Omega)}.$$

LEMMA 4.8. *Let $f \in \mathscr{C}_b(0,T;\mathscr{C}_b^4(\Omega))$, and then*

$$\begin{aligned}&\left\|\mathcal{T}_1^{0,0}\circ\widehat{\mathcal{T}}_2^{0,0}\circ\mathcal{T}_1^{0,0}f(t^n)-\widetilde{\mathcal{T}}_1^{0,0}\circ\widetilde{\mathcal{T}}_2^{0,0}\circ\widetilde{\mathcal{T}}_1^{0,0}f(t^n)\right\|_{L_h^2(\Omega)}\\&\quad\le C\Bigg(\Delta x^4+\Delta v^4+\Delta t\Bigg(\Delta t^2+\Delta x^4+\Delta v^4+\Delta x^5+\Delta x\Delta v^4\\&\quad+\left\|\left(\mathcal{T}_1^{0,0}-\widetilde{\mathcal{T}}_1^{0,0}\right)f(t^n)\right\|_{L_h^2(\Omega)}+\left\|\widetilde{\mathcal{T}}_1^{0,0}\left(f(t^n)-f_h(t^n)\right)\right\|_{L_h^2(\Omega)}\\&\quad+\Delta x\left\|\left(\mathcal{T}_1^{1,0}-\widetilde{\mathcal{T}}_1^{1,0}\right)f(t^n)\right\|_{L_h^2(\Omega)}+\Delta x\left\|\widetilde{\mathcal{T}}_1^{1,0}\left(f(t^n)-f_h(t^n)\right)\right\|_{L_h^2(\Omega)}\Bigg)\Bigg)\end{aligned}$$

and

$$\begin{aligned}&\left\|\nabla\mathcal{T}_1\circ\nabla\widehat{\mathcal{T}}_2\circ\nabla\mathcal{T}_1f(t^n)-\nabla\widetilde{\mathcal{T}}_1\circ\nabla\widetilde{\mathcal{T}}_2\circ\nabla\widetilde{\mathcal{T}}_1f(t^n)\right\|_{\mathbb{L}_h^2(\Omega)}\\&\quad\le C\Bigg(\Delta x^3+\Delta v^3+\Delta t\Bigg(\Delta t^2+\Delta x^3+\Delta x^{-1}\Delta v^4+\Delta v^4+\Delta x^4\\&\quad+\left(1+\Delta x^{-1}\right)\left\|\left(\mathcal{T}_1^{0,0}-\widetilde{\mathcal{T}}_1^{0,0}\right)f(t^n)\right\|_{L_h^2(\Omega)}\\&\quad+\left(1+\Delta x^{-1}\right)\left\|\widetilde{\mathcal{T}}_1^{0,0}\left(f(t^n)-f_h(t^n)\right)\right\|_{L_h^2(\Omega)}\\&\quad+\left\|\left(\mathcal{T}_1^{1,0}-\widetilde{\mathcal{T}}_1^{1,0}\right)f(t^n)\right\|_{L_h^2(\Omega)}+\left\|\widetilde{\mathcal{T}}_1^{1,0}\left(f(t^n)-f_h(t^n)\right)\right\|_{L_h^2(\Omega)}\Bigg)\Bigg).\end{aligned}$$

Proof. First we consider the following decomposition:

$$\begin{aligned}\left\|\mathcal{T}_1^{\alpha}\circ\widehat{\mathcal{T}}_2^{\alpha}\circ\mathcal{T}_1^{\alpha}f(t^n)-\widetilde{\mathcal{T}}_1^{\alpha}\circ\widetilde{\mathcal{T}}_2^{\alpha}\circ\widetilde{\mathcal{T}}_1^{\alpha}f(t^n)\right\|_{L_h^2(\Omega)}&\le\left\|\left(\mathcal{T}_1^{\alpha}-\widetilde{\mathcal{T}}_1^{\alpha}\right)\circ\widehat{\mathcal{T}}_2^{\alpha}\circ\mathcal{T}_1^{\alpha}f(t^n)\right\|_{L_h^2(\Omega)}&(4.31)\\&\quad+\left\|\widetilde{\mathcal{T}}_1^{\alpha}\circ\left(\widehat{\mathcal{T}}_2^{\alpha}-\widetilde{\mathcal{T}}_2^{\alpha}\right)\circ\mathcal{T}_1^{\alpha}f(t^n)\right\|_{L_h^2(\Omega)}&(4.32)\\&\quad+\left\|\widetilde{\mathcal{T}}_1^{\alpha}\circ\widetilde{\mathcal{T}}_2^{\alpha}\circ\left(\mathcal{T}_1^{\alpha}-\widetilde{\mathcal{T}}_1^{\alpha}\right)f(t^n)\right\|_{L_h^2(\Omega)}.&(4.33)\end{aligned}$$

Let us start with the term (4.31). The term (4.31) can be decomposed in the following manner:

$$\begin{aligned}\left\|\left(\mathcal{T}_1^{\alpha}-\widetilde{\mathcal{T}}_1^{\alpha}\right)\circ\widehat{\mathcal{T}}_2^{\alpha}\circ\mathcal{T}_1^{\alpha}f(t^n)\right\|_{L_h^2(\Omega)}&\le\left\|\left(\mathcal{T}_1^{\alpha}-\widetilde{\mathcal{T}}_1^{\alpha}\right)\circ\mathcal{T}_2^{\alpha}\circ\mathcal{T}_1^{\alpha}f(t^n)\right\|_{L_h^2(\Omega)}&(4.34)\\&\quad+\left\|\mathcal{T}_1^{\alpha}\circ\left(\widehat{\mathcal{T}}_2^{\alpha}-\mathcal{T}_2^{\alpha}\right)\circ\mathcal{T}_1^{\alpha}f(t^n)\right\|_{L_h^2(\Omega)}&(4.35)\\&\quad+\left\|\widetilde{\mathcal{T}}_1^{\alpha}\circ\left(\mathcal{T}_2^{\alpha}-\widehat{\mathcal{T}}_2^{\alpha}\right)\circ\mathcal{T}_1^{\alpha}f(t^n)\right\|_{L_h^2(\Omega)}.&(4.36)\end{aligned}$$

Let us proceed with (4.34):

$$\begin{aligned}\left\|\left(\mathcal{T}_1^\alpha-\widetilde{\mathcal{T}}_1^\alpha\right)\circ\mathcal{T}_2^\alpha\circ\mathcal{T}_1^\alpha f(t^n)\right\|_{L_h^2(\Omega)}&\le C\Delta x^{4-|\alpha|}\left\|\mathcal{T}_2^{0,0}\circ\mathcal{T}_1^{0,0}f(t^n)\right\|_{\mathscr{C}_b^4(\Omega)}\\&\le C\left(\|f\|_{\mathscr{C}_b\left(0,T;\mathscr{C}_b^4(\Omega)\right)},\|E\|_{\mathscr{C}_b^4(\Omega)}\right)\Delta x^{4-|\alpha|}\\&\le C\Delta x^{4-|\alpha|}.\end{aligned}$$

The term (4.35) has already been estimated, and the result is given by Lemma 4.5. Let us evaluate the term (4.36). If we introduce the characteristic curves

$$\begin{aligned}\widetilde{X}^\star(t^n)&=x-v\Delta t+\frac{\Delta t^2}{2}E(t^{n+1/2},x),\\\widetilde{V}^\star(t^n)&=v-\Delta tE(t^{n+1/2},x),\\\widehat{X}^\star(t^n)&=x-v\Delta t+\frac{\Delta t^2}{2}E_h^{n+1/2}(x),\\\widehat{V}^\star(t^n)&=v-\Delta tE_h^{n+1/2}(x),\end{aligned}$$

we get

$$\begin{aligned}\left|\left(\left(\widehat{\mathcal{T}}_2^{0,0}-\mathcal{T}_2^{0,0}\right)\circ\mathcal{T}_1^{0,0}f(t^n)\right)_{k,l}\right|&\le\left|f\left(t^n,\widehat{X}_{k,l}^\star(t^n),\widehat{V}_{k,l}^\star(t^n)\right)-f\left(t^n,\widetilde{X}_{k,l}^\star(t^n),\widetilde{V}_{k,l}^\star(t^n)\right)\right|\\&\le C\Delta t\left\|E_h^{n+1/2}-E(t^{n+1/2})\right\|_\infty\|\nabla f\|_{L^\infty(\Omega)}.\end{aligned}$$

We deduce that

$$\left\|\left(\widehat{\mathcal{T}}_2^{0,0}-\mathcal{T}_2^{0,0}\right)\circ\mathcal{T}_1^{0,0}f(t^n)\right\|_{L_h^2(\Omega)}\le C|\Omega|\Delta t\left\|E_h^{n+1/2}-E(t^{n+1/2})\right\|_\infty,$$

and by using (4.25) we finally get

$$\begin{aligned}\left\|\widetilde{\mathcal{T}}_1^{0,0}\circ\left(\widehat{\mathcal{T}}_2^{0,0}-\mathcal{T}_2^{0,0}\right)\circ\mathcal{T}_1^{0,0}f(t^n)\right\|_{L_h^2(\Omega)}&\le C|\Omega|\Delta t\left\|E_h^{n+1/2}-E(t^{n+1/2})\right\|_\infty\\&\le C\Delta t\Bigg(\Delta t^2+\Delta x^4+\Delta v^4\\&\quad+\left\|\left(\mathcal{T}_1^{0,0}-\widetilde{\mathcal{T}}_1^{0,0}\right)f(t^n)\right\|_{L_h^2(\Omega)}\\&\quad+\left\|\widetilde{\mathcal{T}}_1^{0,0}\left(f(t^n)-f_h(t^n)\right)\right\|_{L_h^2(\Omega)}\Bigg).\end{aligned}$$

On the other hand we have

$$\begin{aligned}&\left|\left(\left(\nabla\widehat{\mathcal{T}}_2-\nabla\mathcal{T}_2\right)\circ\nabla\mathcal{T}_1f(t^n)\right)_{k,l}\right|\\&\quad\le\left|\nabla f\left(\widehat{X}_{k,l}^\star(t^n),\widehat{V}_{k,l}^\star(t^n)\right)\right|\cdot\left(\left|\nabla\widehat{X}_{k,l}^\star(t^n)-\nabla\widetilde{X}_{k,l}^\star(t^n)\right|+\left|\nabla\widehat{V}_{k,l}^\star(t^n)-\nabla\widetilde{V}_{k,l}^\star(t^n)\right|\right)\\&\quad+\left|\nabla f\left(\widehat{X}_{k,l}^\star(t^n),\widehat{V}_{k,l}^\star(t^n)\right)-\nabla f\left(\widetilde{X}_{k,l}^\star(t^n),\widetilde{V}_{k,l}^\star(t^n)\right)\right|\cdot\left|\nabla\left(\widetilde{X}_{k,l}^\star(t^n),\widetilde{V}_{k,l}^\star(t^n)\right)\right|\\&\quad\le\left|\nabla f\left(\widehat{X}_{k,l}^\star(t^n),\widehat{V}_{k,l}^\star(t^n)\right)\right|\cdot\left(\left|\nabla\widehat{X}_{k,l}^\star(t^n)-\nabla\widetilde{X}_{k,l}^\star(t^n)\right|+\left|\nabla\widehat{V}_{k,l}^\star(t^n)-\nabla\widetilde{V}_{k,l}^\star(t^n)\right|\right)\\&\quad+\left(\left|\widehat{X}_{k,l}^\star(t^n)-\widetilde{X}_{k,l}^\star(t^n)\right|+\left|\widehat{V}_{k,l}^\star(t^n)-\widetilde{V}_{k,l}^\star(t^n)\right|\right)\\&\qquad\cdot\left|\nabla^2f\left(\widetilde{X}_{k,l}^\star(t^n),\widetilde{V}_{k,l}^\star(t^n)\right)\right|\cdot\left|\nabla\left(\widetilde{X}_{k,l}^\star(t^n),\widetilde{V}_{k,l}^\star(t^n)\right)\right|\end{aligned}$$

$$
\begin{aligned}
&\le C\left(\|\nabla f\|_{\mathscr{C}_b(0,T;\mathscr{C}_b(\Omega))}, \left\|\nabla^2 f\right\|_{\mathscr{C}_b(0,T;\mathscr{C}_b(\Omega))}, \|\partial_x E\|_{\mathscr{C}_b(0,T;\mathscr{C}_b(\Omega))}\right)\\
&\times\left(\left|\widehat{X}^\star_{k,l}(t^n)-\widetilde{X}^\star_{k,l}(t^n)\right|+\left|\widehat{V}^\star_{k,l}(t^n)-\widetilde{V}^\star_{k,l}(t^n)\right|\right.\\
&\left.+\left|\nabla\widehat{X}^\star_{k,l}(t^n)-\nabla\widetilde{X}^\star_{k,l}(t^n)\right|+\left|\nabla\widehat{V}^\star_{k,l}(t^n)-\nabla\widetilde{V}^\star_{k,l}(t^n)\right|\right)\\
&\le C\Delta t\left\|E_h^{n+1/2}-E(t^{n+1/2})\right\|_{1,\infty}.
\end{aligned}
$$

From inequalities (4.26), (4.27), (4.29), and (4.30) we find that

$$
\begin{aligned}
&\left\|\nabla\widetilde{\mathcal{T}}_1\left(\nabla\widehat{\mathcal{T}}_2-\nabla\mathcal{T}_2\right)\circ\nabla\mathcal{T}_1 f(t^n)\right\|_{\mathbb{L}^2_h(\Omega)}\le(1+C\Delta t)\left\|\left(\nabla\widehat{\mathcal{T}}_2-\nabla\mathcal{T}_2\right)\circ\nabla\mathcal{T}_1 f(t^n)\right\|_{\mathbb{L}^2_h(\Omega)}\\
&\quad\le C|\Omega|\Delta t\left\|E_h^{n+1/2}-E(t^{n+1/2})\right\|_{1,\infty}\\
&\quad\le C\Delta t\left(\Delta t^2+\Delta x^4+\Delta v^4+\left\|\left(\mathcal{T}_1^{0,0}-\widetilde{\mathcal{T}}_1^{0,0}\right)f(t^n)\right\|_{L^2_h(\Omega)}\right.\\
&\quad+\left\|\widetilde{\mathcal{T}}_1^{0,0}\left(f(t^n)-f_h(t^n)\right)\right\|_{L^2_h(\Omega)}+\left\|\left(\mathcal{T}_1^{1,0}-\widetilde{\mathcal{T}}_1^{1,0}\right)f(t^n)\right\|_{L^2_h(\Omega)}\\
&\quad\left.+\left\|\widetilde{\mathcal{T}}_1^{1,0}\left(f(t^n)-f_h(t^n)\right)\right\|_{L^2_h(\Omega)}\right).
\end{aligned}
$$

Let us continue with the term (4.32). By using (4.25) we get that

$$
\begin{aligned}
\left\|\widetilde{\mathcal{T}}_1^{0,0}\circ\left(\widehat{\mathcal{T}}_2^{0,0}-\widetilde{\mathcal{T}}_2^{0,0}\right)\circ\mathcal{T}_1^{0,0}f(t^n)\right\|_{L^2_h(\Omega)}&\le\left\|\left(\widehat{\mathcal{T}}_2^{0,0}-\widetilde{\mathcal{T}}_2^{0,0}\right)\circ\mathcal{T}_1^{0,0}f(t^n)\right\|_{L^2_h(\Omega)}\\
&\le C\Delta v^4\left\|\mathcal{T}_1^{0,0}f(t^n)\right\|_{\mathscr{C}^4_b(\Omega)}\\
&\le C\Delta v^4\|f\|_{\mathscr{C}\left(0,T;\mathscr{C}^4_b(\Omega)\right)}.
\end{aligned}
$$

By using (4.26) and (4.27) we have

$$
\begin{aligned}
\left\|\nabla\widetilde{\mathcal{T}}_1\circ\left(\nabla\widehat{\mathcal{T}}_2-\nabla\widetilde{\mathcal{T}}_2\right)\circ\nabla\mathcal{T}_1 f(t^n)\right\|_{\mathbb{L}^2_h(\Omega)}&\le(1+C\Delta t)\left\|\left(\nabla\widehat{\mathcal{T}}_2-\nabla\widetilde{\mathcal{T}}_2\right)\circ\nabla\mathcal{T}_1 f(t^n)\right\|_{\mathbb{L}^2_h(\Omega)}\\
&\le C\Delta v^3\left\|\mathcal{T}_1^{0,0}f(t^n)\right\|_{\mathscr{C}^4_b(\Omega)}\\
&\le C\Delta v^3\|f\|_{\mathscr{C}\left(0,T;\mathscr{C}^4_b(\Omega)\right)}.
\end{aligned}
$$

Let us finish with the term (4.33). By using (4.25) and (4.28) we get that

$$
\begin{aligned}
\left\|\widetilde{\mathcal{T}}_1^{0,0}\circ\widetilde{\mathcal{T}}_2^{0,0}\circ\left(\mathcal{T}_1^{0,0}-\widetilde{\mathcal{T}}_1^{0,0}\right)f(t^n)\right\|_{L^2_h(\Omega)}&\le\left\|\widetilde{\mathcal{T}}_2^{0,0}\circ\left(\mathcal{T}_1^{0,0}-\widetilde{\mathcal{T}}_1^{0,0}\right)f(t^n)\right\|_{L^2_h(\Omega)}\\
&\le\left\|\left(\mathcal{T}_1^{0,0}-\widetilde{\mathcal{T}}_1^{0,0}\right)f(t^n)\right\|_{L^2_h(\Omega)}\\
&\le C\Delta x^4\left\|\mathcal{T}_1^{0,0}f(t^n)\right\|_{\mathscr{C}^4_b(\Omega)}\\
&\le C\Delta x^4\|f\|_{\mathscr{C}\left(0,T;\mathscr{C}^4_b(\Omega)\right)}.
\end{aligned}
$$

By using (4.26), (4.27), (4.29), and (4.30) we get that

$$\begin{aligned}\left\|\nabla\widetilde{\mathcal{T}}_1\circ\nabla\widetilde{\mathcal{T}}_2\circ\left(\nabla\mathcal{T}_1-\nabla\widetilde{\mathcal{T}}_1\right)f(t^n)\right\|_{\mathbb{L}^2_h(\Omega)} &\le (1+C\Delta t)\left\|\nabla\widetilde{\mathcal{T}}_2\circ\left(\nabla\mathcal{T}_1-\nabla\widetilde{\mathcal{T}}_1\right)f(t^n)\right\|_{\mathbb{L}^2_h(\Omega)}\\ &\le (1+C\Delta t)^2\left\|\left(\nabla\mathcal{T}_1-\nabla\widetilde{\mathcal{T}}_1\right)f(t^n)\right\|_{\mathbb{L}^2_h(\Omega)}\\ &\le C\Delta x^3\left\|f(t^n)\right\|_{\mathscr{C}^4_b(\Omega)}\\ &\le C\Delta x^3\left\|f\right\|_{\mathscr{C}\left(0,T;\mathscr{C}^4_b(\Omega)\right)}.\qquad\square\end{aligned}$$

Now let us evaluate the term (I4).

LEMMA 4.9. *Let $f\in\mathscr{C}_b(0,T;\mathscr{C}^1_b(\Omega))$, and then there exists a constant C independent of Δx, Δv, and Δt such that*

$$\left\|\widetilde{\mathcal{T}}^{0,0}_1\circ\widetilde{\mathcal{T}}^{0,0}_2\circ\widetilde{\mathcal{T}}^{0,0}_1(f(t^n)-f_h(t^n))\right\|_{L^2_h(\Omega)}\le\left\|f(t^n)-f_h(t^n)\right\|_{L^2_h(\Omega)}$$

and

$$\left\|\nabla\widetilde{\mathcal{T}}_1\circ\nabla\widetilde{\mathcal{T}}_2\circ\nabla\widetilde{\mathcal{T}}_1(f(t^n)-f_h(t^n))\right\|_{\mathbb{L}^2_h(\Omega)}\le(1+C\Delta t)\left\|\nabla f(t^n)-\nabla f_h(t^n)\right\|_{\mathbb{L}^2_h(\Omega)}.$$

Proof. By using (4.25) and (4.28) we get that

$$\begin{aligned}\left\|\widetilde{\mathcal{T}}^{0,0}_1\circ\widetilde{\mathcal{T}}^{0,0}_2\circ\widetilde{\mathcal{T}}^{0,0}_1(f(t^n)-f_h(t^n))\right\|_{L^2_h(\Omega)} &\le\left\|\widetilde{\mathcal{T}}^{0,0}_2\circ\widetilde{\mathcal{T}}^{0,0}_1(f(t^n)-f_h(t^n))\right\|_{L^2_h(\Omega)}\\ &\le\left\|\widetilde{\mathcal{T}}^{0,0}_1(f(t^n)-f_h(t^n))\right\|_{L^2_h(\Omega)}\\ &\le\left\|f(t^n)-f_h(t^n)\right\|_{L^2_h(\Omega)}.\end{aligned}$$

By using (4.26), (4.27), (4.29), and (4.30) it follows that

$$\begin{aligned}&\left\|\nabla\widetilde{\mathcal{T}}_1\circ\nabla\widetilde{\mathcal{T}}_2\circ\nabla\widetilde{\mathcal{T}}_1(f(t^n)-f_h(t^n))\right\|_{\mathbb{L}^2_h(\Omega)}\\ &\qquad\le(1+C\Delta t)\left\|\nabla\widetilde{\mathcal{T}}_2\circ\nabla\widetilde{\mathcal{T}}_1(f(t^n)-f_h(t^n))\right\|_{\mathbb{L}^2_h(\Omega)}\\ &\qquad\le(1+C\Delta t)^2\left\|\nabla\widetilde{\mathcal{T}}_1(f(t^n)-f_h(t^n))\right\|_{\mathbb{L}^2_h(\Omega)}\\ &\qquad\le(1+C\Delta t)^3\left\|\nabla f(t^n)-\nabla f_h(t^n)\right\|_{\mathbb{L}^2_h(\Omega)}\\ &\qquad\le(1+C\Delta t)\left\|\nabla f(t^n)-\nabla f_h(t^n)\right\|_{\mathbb{L}^2_h(\Omega)}.\qquad\square\end{aligned}$$

Proof of Theorem 4.1. By putting together Lemmas 4.3, 4.5, 4.7, 4.8, and 4.9 we obtain

$$\begin{pmatrix}\|e_{n+1}\|_{L^2_h(\Omega)}\\ \|\nabla e_{n+1}\|_{\mathbb{L}^2_h(\Omega)}\end{pmatrix}=\begin{pmatrix}1+C\Delta t & \Delta t\Delta x\\ \Delta t\left(1+\Delta x^{-1}\right) & 1+C\Delta t\end{pmatrix}\begin{pmatrix}\|e_n\|_{L^2_h(\Omega)}\\ \|\nabla e_n\|_{\mathbb{L}^2_h(\Omega)}\end{pmatrix}+\begin{pmatrix}\gamma\\ \partial\gamma\end{pmatrix},$$

where

$$\gamma=C\left(\Delta t^3+(1+\Delta t+\Delta x)(\Delta x^4+\Delta v^4)+\Delta x^4+\Delta x^4\right)$$

and

$$\partial\gamma=C\left(\Delta t^3+\Delta x^{-1}(1+\Delta t+\Delta x)(\Delta x^4+\Delta v^4)+\Delta x^3+\Delta x^3\right).$$

The eigenvalues of the matrix

$$\begin{pmatrix} 1 + C\Delta t & \Delta t \Delta x \\ \Delta t \left(1 + \Delta x^{-1}\right) & 1 + C\Delta t \end{pmatrix}$$

are equivalent to $1 + \mathcal{O}(\Delta t)$. The formula of recurrence gives

$$\begin{pmatrix} \|e_{n+1}\|_{L_h^2(\Omega)} \\ \|\nabla e_{n+1}\|_{\mathbb{L}_h^2(\Omega)} \end{pmatrix} = \frac{1}{\Delta t} \begin{pmatrix} \gamma \\ \partial \gamma \end{pmatrix} + e^{CT} \begin{pmatrix} \|e_0\|_{L_h^2(\Omega)} \\ \|\nabla e_0\|_{\mathbb{L}_h^2(\Omega)} \end{pmatrix},$$

which is the error estimate of $\partial^\alpha f - \partial^\alpha f_h$ stated in Theorem 4.1, assuming $\|e_0\|_{L_h^2(\Omega)} = \|\nabla e_0\|_{\mathbb{L}_h^2(\Omega)} = 0$. By using the estimates (4.23) and (4.24) and thanks to the error estimate of $\partial^\alpha f - \partial^\alpha f_h$, we find the estimate $\partial^\alpha E - \partial^\alpha E_h$ stated in Theorem 4.1. □

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