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Quasilinear analysis of the gyro-water-bag model

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Abstract – The energy confinement time in controlled-fusion devices is governed by the turbulent evolution of low-frequency electromagnetic fluctuations of nonuniform magnetized plasmas. The necessary kinetic calculation of turbulent transport consumes much more computer resources than fluid simulations. An alternative approach is based on water-bag–like weak solution of collisionless kinetic equations, allowing to reduce the Vlasov equation into a set of hyrodynamic equations while keeping its kinetic behaviour. In this paper we apply this concept to gyrokinetic modeling, and focus on the weak turbulence theory of the gyro-water-bag model. As a result we obtain a set of nonlinear diffusion equations where the source terms are the divergence of the parallel fluctuating Reynolds stress of each bag. These source terms describe the process of correlated radial scattering and parallel acceleration which is required to generate a sheared parallel flow and may have important consequences for the theory of both intrinsic rotation and momentum transport bifurcations which are closely related to confinement improvements and internal transport barrier dynamics in tokamaks. Using the kinetic resonance condition our quasilinear equations can be recast in a model whose the mathematical structure is the same as the famous Keller-Segel model, widely used in chemotaxis to describe the collective transport (diffusion and aggregation) of cells attracted by a self-emitted chemical substance. Therefore the second result of the paper is the derivation of a set of reaction-diffusion equations which describes the interplay between the turbulence process in the radial direction and the back reaction of the zonal flow in the poloidal direction.

A statistical approach of turbulence is always attractive because it allows to get transport coefficients without computing the full fluctuations. In turbulence of toroidal magnetically confined plasmas these transport coefficients are crucial to determine the time confinement and scaling laws. Moreover, predicting turbulence in nearly collisionless fusion plasmas requires to solve Vlasov gyrokinetic equations coupled to low-frequency approximations of Maxwell equations, because fluid description (MHD), even useful and quite relevant for the most physical phenomena, do not take into account the resonant wave-particle interaction which plays an important role in the quantization of the turbulent diffusion. Nevertheless, the drawback of kinetic models is the large dimensionality of the phase space, resulting in the need of very large computer resources. For example the gyrokinetic equations evolve in a four-dimensional phase-space, plus the adiabatic-invariant (magnetic-moment) one-dimensional space which only needs a coarse sampling because of the invariance property.

Here, we introduce another invariant, this time being exact and named "water-bag". This exact invariant comes from the Liouville theorem which expresses the fact that the phase-space volume measure is preserved during time. Indeed the water-bag model, introduced by DePackh [1], Hohl, Feix and Bertrand [2,3], was shown to bring the bridge between the fluid and the kinetic description of a collisionless plasma, allowing to keep the kinetic aspect of the problem with the same complexity as a multi-fluid model. The gyro-water-bag is born from these two phase space variable reductions through the existence of two underlying invariants, one being adiabatic and the other being exact [4]. From a theoretical point of view this model is very interesting because it has a simple fluid-like structure allowing to pursue analysis far enough. From the numerical point of view the interest of this model is obvious because it yields an additional reduction, resulting in less expensive algorithms than the ones deduced from

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the kinetic description. This last remark will be considered in a forthcoming paper.

The aim of this paper is to introduce the gyro-water-bag for gyrokinetic modeling in the frame of weak turbulence theory. The important result of the present paper is the derivation of a set of coupled nonlinear diffusion equations where source terms are the fluctuation-driven parallel Reynolds stress $\langle \tilde{v}_{r_{E\times B}} \tilde{v}_{\|} \rangle$. These terms are proportional to the correlation $\langle \nabla_\theta \tilde{\phi} \nabla_{\|} \tilde{\phi} \rangle$, which is responsible for regulating the radial transport of parallel momentum. These non-diffusive, off-diagonal terms contribute to the momentum flux which may account for the radially inward —momentum pinch— (or outward) transport of parallel momentum and the spontaneous toroidal rotation resulting in a peaked rotation profile on the axis. There is great interest in discovering a possible fluctuation-driven momentum pinch —inward stress— and relating it to the overall profile self-organization structure such as internal transport barriers. Some studies of fluctuation-driven momentum pinch have been carried out in a fluid framework [5] where the effect of a sheared toroidal velocity on internal transport barriers is analyzed and in a Vlasovian kinetic framework [6] where a unified approach of the theory of turbulent transport of parallel momentum by collisionless drift waves is performed. Secondly, using the kinetic resonance condition, the quasilinear equations discussed above can be recast into a set of coupled reaction-diffusion equations which describes the competition between the turbulent diffusion in the radial direction and the back reaction of self-generated zonal flow in the poloidal direction which may reduce the turbulent state and lead to the formation of internal transport barriers. The mathematical structure of our equations is the same as the famous Keller-Segel model [7], widely used in chemotaxis models to describe the collective transport (diffusion, concentration and aggregation) of cells —of a biological multicellular organism— attracted by a self-emitted chemical substance (chemo-attractant). In this picture the bags play the role of the density of different cell groups and the zonal flow, through the mean electrical potential, plays the role of the chemo-attractant.

The water-bag approach in the gyrokinetic theory. – The gyrokinetic approach has been widely used in recent years to study low-frequency micro-instabilities in a magnetically-confined plasma which are known for exibiting a wide range of spatial and temporal scales. The conventional procedure [8], to derive the gyrokinetic Vlasov equations consists in computing an iterative solution of the gyroangle-averaged Vlasov equation perturbatively expanded in powers of a dimensionless parameter ρ/L, where ρ is the Larmor radius and where L represents the characteristic background magnetic-field, plasma density and temperature nonuniformity length scale. A modern foundation of the nonlinear gyrokinetic theory [9–11] is based on two-step Lie-transform procedures from particles Hamiltonian dynamics to gyrocenter motion through guiding-center dynamics and a reduced variational principle [11,12] allowing to derive self-consistent expressions for the nonlinear gyrokinetic Vlasov Maxwell equations. In the sequel we deal with a simpler model than the general equations presented in [9–11] by making the following approximations. In addition to the cylindrical geometry assumption, we suppose that the magnetic field $\mathbf{B}$ is uniform and constant along the axis of the column (z-coordinate, $\mathbf{B}=B\mathbf{b}=B\mathbf{e}_z$). As a result the perpendicular drift velocity does not admit any magnetic curvature or gradient effect. It is important to point out that the water-bag concept (*i.e.*, phase space conservation), that we will expose below, is not affected by adding curvature terms, excepting of course a more complicated algebra. We next linearize the expression of the polarization density by neglecting all terms smaller than $\mathcal{O}(k_\perp^2\rho_i^2)$ and we assume that the ion cyclotron frequency, Ω_i, is a constant. Moreover we assume that the ion Debye lenght λ_{D_i} is small compared to the ion Larmor radius ρ_i. Finally the electron inertia is ignored, *i.e.* we choose an adiabatical response to the low-frequency fluctuations for the electrons. In other words the electrons density follows the Boltzmann distribution $n_e=n_{e0}\exp\left(e(\phi-\lambda\langle\phi\rangle_{\mathcal{M}})/(k_BT_e)\right)$, where $\langle\phi\rangle_{\mathcal{M}}$ denotes the average of the electrical potential ϕ over a magnetic surface. The parameter λ is a control parameter for zonal flows and takes the value zero or one. We denote $\mathcal{J}_\mu$ the linear operator of gyroaveraging with $\mu=m_iv_\perp^2/(2B)$ the adiabatic invariant. As $\mathcal{J}_\mu$ is a convolution operator, in Fourier space it becomes a multiplication operator by introducing the Bessel function of the first kind $J_0(k_\perp\rho_i)=J_0(k_\perp\sqrt{2\mu/(\Omega_iq_i)})$. We suppose a finite discrete sequence of adiabatic invariant $\Lambda=\{\mu\}$ linked to a finite discrete sequence of ion Larmor radius $\Upsilon=\{\rho\}$ by $\mu=\rho^2\Omega_iq_i/2$. The structure of the distribution function f, solution of the gyrokinetic Vlasov equation, is of the form $f(t,\mathbf{r},v_\|,\mu)=\sum_{\eta\in\Lambda}f_\eta(t,\mathbf{r},v_\|)\delta(\mu-\eta)$. Let us notice that an interesting problem is to know what is the asymptotic statistical distribution function in μ if we consider an infinite number of magnetic moments in the previous sum, because it allows to save CPU time and memory space in numerical codes. Under these assumptions the evolution of the ion gyrocenter distribution function $f_\mu=f_\mu(t,\mathbf{r}_\perp,z,v_\|)$ obeys the gyrokinetic Vlasov equation

$$\partial_t f_\mu + \frac{\mathcal{J}_\mu\mathbf{E}\times\mathbf{B}}{B^2}\cdot\nabla_\perp f_\mu + v_\|\partial_z f_\mu + \frac{q_i}{m_i}\mathcal{J}_\mu E_\|\partial_{v_\|}f_\mu = 0 \tag{1}$$

for the ions (q_i,M_i), coupled to an adiabatic electron response via the quasineutrality assumption ($\lambda\in\{0,1\}$)

$$-\nabla_\perp\cdot\left(\frac{n_{i0}}{B\Omega_i}\nabla_\perp\phi\right)+\frac{en_{i0}}{k_BT_e}(\phi-\lambda\langle\phi\rangle_{\mathcal{M}}) = 2\pi\sum_{\mu\in\Lambda}\int_{\mathbb{R}}\frac{\Omega_i}{q_i}\mathcal{J}_\mu f_\mu(t,\mathbf{r},v_\|)\mathrm{d}v_\| - n_{i0}. \tag{2}$$

The most important and interesting feature is that f_μ depends, through a differential operator, only on one velocity component, $v_\parallel$, parallel to $\mathbf{B}$.

Let us now turn back to the gyrokinetic equation (1) and for each $\mu \in \Lambda$ let us consider $2\mathcal{N}$ non-closed contours in the $(\mathbf{r}, v_\parallel)$-phase space labelled $v^+_{\mu j}$ and $v^-_{\mu j}$ (where $j = 1, \ldots, \mathcal{N}$, $\mu \in \Lambda$) such that $\ldots < v^-_{\mu j+1} < v^-_{\mu j} < \ldots < 0 < \ldots < v^+_{\mu j} < v^+_{\mu j+1} < \ldots$ and some constant real number $\{\mathcal{A}_{\mu j}\}_{j\in[1,\mathcal{N}],\mu\in\Lambda}$ that we call bag heights. Then we define $f_\mu(t, \mathbf{r}_\perp, z, v_\parallel)$ as

$$f_\mu(\mathbf{t}, \mathbf{r}_\perp, z, v_\parallel) = \sum_{j=1}^{\mathcal{N}} \mathcal{A}_{\mu j} \left[\mathcal{H}\left(v_\parallel - v^-_{\mu j}(t, \mathbf{r}_\perp, z)\right) - \mathcal{H}\left(v_\parallel - v^+_{\mu j}(t, \mathbf{r}_\perp, z)\right)\right], \qquad (3)$$

where $\mathcal{H}$ denotes the Heaviside unit step function. The function (3) is an *exact weak* solution of the gyrokinetic Vlasov equation (1) in the sense of distribution theory, if and only if the set of the following equations is satisfied:

$$\partial_t v^\pm_{\mu j} + \mathcal{J}_\mu \mathbf{v_E}.\nabla_\perp v^\pm_{\mu j} + v^\pm_{\mu j}\partial_z v^\pm_{\mu j} = \frac{q_i}{m_i}\mathcal{J}_\mu E_\parallel, \qquad (4)$$

for all $j \in [1, \mathcal{N}]$ and $\mu \in \Lambda$. The source term in the quasineutrality equation (2) can be rewritten as $2\pi \sum_{\mu\in\Lambda}\sum_{j=1}^{\mathcal{N}} \frac{\Omega_i}{q_i}\mathcal{A}_{\mu j}\mathcal{J}_\mu(v^+_{\mu j} - v^-_{\mu j}) - n_{i0}$.

It is illuminating to introduce for each bag j the density $n_{\mu j} = (v^+_{\mu j} - v^-_{\mu j})\mathcal{A}_{\mu j}$ and the average velocity $u_{\mu j} = (v^+_{\mu j} + v^-_{\mu j})/2$. After little algebra, eqs. (4), lead to continuity and Euler equations namely

$$\partial_t n_{\mu j} + \nabla_\perp \cdot (n_{\mu j}\mathcal{J}_\mu \mathbf{v_E}) + \partial_z(n_{\mu j}u_{\mu j}) = 0,$$

$$\partial_t(n_{\mu j}u_{\mu j}) + \nabla_\perp \cdot (n_{\mu j}u_{\mu j}\mathcal{J}_\mu \mathbf{v_E}) + \partial_z\left(n_{\mu j}u^2_{\mu j}\right) + \frac{1}{m_i}\partial_z p_{\mu j} = \frac{q_i}{m_i} n_{\mu j}\mathcal{J}_\mu E_\parallel,$$

where the partial pressure takes the form $p_{\mu j} = m_i n^3_{\mu j}/(12\mathcal{A}^2_{\mu j})$. The connection between the kinetic and the fluid description clearly appears in the previous multi-fluid equations. The case of one bag recovers a fluid description (with an exact adiabatic closure with $\gamma = 3$). Consequently, the gyro-water-bag provides a fully kinetic description which is shown to be equivalent to a multi-fluid one. Actually each bag is a fluid described by Euler's equations with a specific adiabatic closure while the coupling between all the fluids is given by the quasineutrality equation (2). The sum over the bags in the source term of (2) allows to recover the kinetic character (nonlinear resonant wave-particle interaction) from a set of fluid equations. For example, in the more simple electrostatic Vlasov-Poisson plasmas [3,4], the well-known Landau damping is recovered by a phase-mixing process of $\mathcal{N}$ discrete undamped fluid eigenmodes, which is reminiscent of the Van Kampen-Case [13,14] treatment of electronic plasma oscillations. Indeed Van Kampen and Case [13,14] have shown that a stable plasma has infinite real eigenvalues (frequencies) whose spectrum is dense on the real axis. By a phase-mixing process, the superposition of these real frequencies, leads to the Landau effect. In the same way when the number of bags $\mathcal{N}$ increases, the real frequencies (eigenvalues) of the water-bag model become dense on the real axis. Therefore the water-bag model allows to recover the Landau effect by the phase mixing of $\mathcal{N}$ undamped eigenmodes. The water-bag eigenmodes can be viewed as a "discrete" version of the Van Kampen eigenmodes.

In eqs. (4), j is nothing but a *label* since *no* differential operation is carried out on the variable $v_\parallel$. What we actually do is to bunch together particles within the same bag j and let each bag evolve using the contour equations (4). Of course, the different bags are coupled through the quasineutrality equation. This operation appears as an *exact* reduction of the phase space dimension (elimination of the velocity variable) in the sense that the water-bag concept makes full use of the Liouville invariance in phase space: the fact that $\mathcal{A}_{\mu j}$'s are constant in time is nothing but a straightforward consequence of the Vlasov conservation $Df/Dt = 0$. Of course, the eliminated velocity reappears in the various bags j ($j = 1, \ldots, \mathcal{N}$), and if we need a precise description of a continuous distribution, a large $\mathcal{N}$ is needed. On the other hand, there is no mathematical lower bound on $\mathcal{N}$ and from a physical point of view many interesting results can even be obtained with $\mathcal{N}$ as small as 1 for electrostatic plasmas. For magnetized plasmas, $\mathcal{N} = 2$ or 3 allow more analytical approaches [4].

On the contrary, in the Vlasov phase space $(\mathbf{r}, v_\parallel)$, the exchange of velocity is described by a differential operator. From a numerical point of view, this operator has to be *approximated* by some finite-difference scheme. Consequently, a minimum size for the mesh in the velocity space is required and we are faced with the usual sampling problem. If it can be claimed that the $v_\parallel$-gradients of the distribution function remain weak enough for some class of problems, then a rough sampling might be acceptable. However, it is well known in kinetic theory that wave-particle interaction is often not so obvious. For instance, steep gradients in the velocity space can be the signature of strong wave-particle interaction and there is the need for a higher numerical resolution of the Vlasov code, while a water-bag description can still be used with a small bag number.

To conclude, the gyro-water-bag offers an exact description of the plasma dynamics even with a small bag number, allowing more analytical studies and bringing the link between the hydrodynamic description and the full Vlasov one. Of course this needs a special initial preparation of the plasma (Lebesgue subdivision). Futhermore, there is no constraint on the shape of the distribution function which can be very far from a Maxwellian one. Once initial data has been prepared using Lebesgue subdivision, the gyro-water-bag equations give the *exact weak* (in the sense of the theory of distribution) solution of the Vlasov equation corresponding to this initial data. Any initial condition (continuous or not) which is integrable with respect to the Lebesgue measure can be approximated accurately with larger $\mathcal{N}$. Therefore, if we need a precise description of a continuous distribution, it is clear that a larger $\mathcal{N}$ is needed; but even if the numerical effort is closed to a standard discretization of the velocity space in a regular Vlasov code (using $2\mathcal{N}+1$ mesh points), we believe that the use of an exact water-bag sampling should give better results than approximating the corresponding differential operator.

Weak-turbulence theory of the gyro-water-bag model. – It is important to point out that the water-bag concept (*i.e.*, phase space conservation) is not affected by adding curvature terms, excepting of course a more complicated algebra. In order to make the quasilinear analysis [15–18] of eqs. (4) and (2) each physical quantity $f\in\{v^{\pm}_{\mu j},\phi\}$ is expanded as a sum of a slowly time-evolving (θ,z)-uniform state f_0 and a small fisrt-order fluctuating perturbation δf such that $f(t,r,\theta,z)=f_0(t,r)+\delta f(t,r,\theta,z)$, where δf is Fourier expanded as $\delta f(t,r,\theta,z)=\sum_{\nu\neq\mathbf{0}} f_\nu(t,r)\exp(i(k_\parallel z+m\theta))$ with $k_\parallel=2\pi n/L_z$, $k_\theta=m/r$, $\nu=(m,n)$ and $\mathbf{0}=(0,0)$. Let us note that $\langle\delta f\rangle_{\theta,z}=0$, $\langle f_0\rangle_{\theta,z}=f_0$, where $\langle\cdot\rangle_{\theta,z}$ denotes (θ,z)-averaging, and $f_{-\nu}=\bar{f}_\nu$. Let us first find the equations for the slowly evolving part $(\{v^{\pm}_{\mu j0}\}_{j\in[1,\mathcal{N}],\,\mu\in\Lambda},\phi_0)$. If we perform the (θ,z)-average of eqs. (4), using Fourier expansion, we get

$$\partial_t v^{\pm}_{\mu j0}+\frac{i}{rB}\partial_r\sum_{\nu\neq\mathbf{0}} rk_\theta J_{0\mu\nu}\bar{\phi}_\nu v^{\pm}_{\mu j\nu}=0, \tag{5}$$

where we have used the notation $J_{0\mu\nu}=J_0(k_\perp\sqrt{2\mu/(\Omega_i q_i)})$. If we substract (5) to (4) and neglect the second-order term in the perturbation, after integrating in time we get the following equations for the first-order perturbation part:

$$\begin{aligned}v^{\pm}_{\mu j\nu}(t)&=v^{\pm}_{\mu j\nu}(0)\exp\left(-i\int_0^t \mathbf{k}\cdot\mathbf{V}^{\pm}_{\mu j0}(s)\mathrm{d}s\right)-i\int_0^t\mathrm{d}s\\&\times J_{0\mu\nu}\phi_\nu(t-s)\lambda^{\pm}_{\mu j0}(t-s)\exp\left(-i\int_{t-s}^t\mathbf{k}\cdot\mathbf{V}^{\pm}_{\mu j0}(\tau)\mathrm{d}\tau\right),\end{aligned} \tag{6}$$

where $\mathbf{k}=(k_\parallel,k_\theta)^T$, $\mathbf{V}^{\pm}_{\mu j0}=(v^{\pm}_{\mu j0},\partial_r\phi_0/B)^T$, and $\lambda^{\pm}_{\mu j0}=(q_i/m_i)k_\parallel-k_\theta\partial_r v^{\pm}_{\mu j0}/B$. Note that eq. (14) for the slow part ϕ_0 and the equation for the first-order perturbation mode ϕ_ν are obtained straightforwardly from the quasineutrality equation (2). We now look for solution of the form (WKB ansatz) $f_\nu(t,r)=\tilde{f}_\nu(t,r)\exp(-i\omega_\nu t)$, where the phase $\omega_\nu=\omega_{\mathbf{k}}$ is a real number such that $\omega_{-\nu}=\omega_{-\mathbf{k}}=-\omega_{\mathbf{k}}=-\omega_\nu$, and where the envelope $\tilde{f}_\nu(t)$ is a time slowly varying real function on a time scale like $\gamma_{\mathbf{k}}^{-1}$. Here $\gamma_{\mathbf{k}}$ is the growth rate of the instability and thus it gives the time scale of the evolution of ϕ_ν, while τ_{dif} is the time scale of the variation of the slowly evolving part $(\{v^{\pm}_{\mu j0}\}_{j\in[1,\mathcal{N}],\,\mu\in\Lambda},\phi_0)$ (due to the diffusion). Since the time scale of the fluctuating envelope is greater than the time scale of the fluctuation oscillation $\tau_{fo}\sim\omega_{\mathbf{k}}^{-1}$, we have $|\gamma_{\mathbf{k}}/\omega_{\mathbf{k}}|\ll 1$. For the sake of clarity we drop the tilde notation in the sequel of the paper. The first term of the right-hand side of (6), the free-streaming part of the solution, will rapidly damp because of the phase mixing in the j-summation or $v_\parallel$-integration on a time $\tau_d\sim(k_\parallel\bar{v})^{-1}$, where $\bar{v}$ is the characteristic spread in the parallel velocity, *i.e.* the parallel thermal velocity. Assuming that $t\gg\tau_d$, we can drop the initial-value term of (6) and substitute it to the source of the equation of the first-order perturbation mode ϕ_ν to obtain

$$-\chi_0\partial_r\Big(\frac{\partial_r\phi_\nu}{\chi_0}\Big)+K\phi_\nu=-iL\int_0^t C(t,s)\phi_\nu(t-s)\mathrm{d}s, \tag{7}$$

where $K=m^2/r^2+\kappa$, $\kappa=eB\Omega_i/(k_BT_e)$, $L=2\pi\Omega_i^2 B/(q_i n_{i0})$, $\chi_0=1/(rn_{i0})$, and

$$\begin{aligned}C(t,s)=\sum_{\substack{\mu\in\Lambda\\ j\leqslant\mathcal{N}}}\mathcal{A}_{\mu j}J^2_{0\mu\nu}\Bigg\{&\lambda^{+}_{\mu j0}(t-s)\exp\left(i\int_{t-s}^t\Omega^{+}_{\mu j0}(\tau)\mathrm{d}\tau\right)\\&-\lambda^{-}_{\mu j0}(t-s)\exp\left(i\int_{t-s}^t\Omega^{-}_{\mu j0}(\tau)\mathrm{d}\tau\right)\Bigg\}\end{aligned}$$

with $\Omega^{\pm}_{\mu j0}(\tau)=\omega_{\mathbf{k}}-\mathbf{k}\cdot\mathbf{V}^{\pm}_{\mu j0}(\tau)$ the Doppler-shifted pulsation. Once again by phase-mixing arguments, the sum over j leads to the decay of $C(t,s)$ with s on a time τ_d. If this time is short compared to τ_{dif} and $\gamma_{\mathbf{k}}^{-1}$, we may do a Taylor expansion of $\lambda^{\pm}_{\mu j0}$, $\mathbf{V}^{\pm}_{\mu j0}$, and ϕ_ν in time in (7). Therefore, if we neglect all terms of second-order in time and assuming $\tau_{dif}\gg\gamma_{\mathbf{k}}^{-1}$, a time-integration yields

$$\begin{aligned}&\dot{\phi}_\nu L\sum_{\mu,j}\mathcal{A}_{\mu j}J^2_{0\mu\nu}\partial_\omega\left(\lambda^{+}_{\mu j0}\delta^{+}_{+\mu j}-\lambda^{-}_{\mu j0}\delta^{-}_{+\mu j}\right)+\chi_0\partial_r\left(\frac{\partial_r\phi_\nu}{\chi_0}\right)\\&-\left\{K+iL\sum_{\mu,j}\mathcal{A}_{\mu j}J^2_{0\mu\nu}\left(\lambda^{+}_{\mu j0}\delta^{+}_{+\mu j}-\lambda^{-}_{\mu j0}\delta^{-}_{+\mu j}\right)\right\}\phi_\nu=0,\end{aligned} \tag{8}$$

where we have used the notations $\dot{\phi}_\nu=\mathrm{d}\phi_\nu/\mathrm{d}t$, $\delta^{\pm}_{+\mu j}=\delta_+(\omega_{\mathbf{k}}-\mathbf{k}\cdot\mathbf{V}^{\pm}_{\mu j0})$ and with $\delta_+(\cdot)=\pi\delta(\cdot)+i\mathrm{p.v.}(1/\cdot)$. The imaginary part of (8) gives the growth rate $\gamma_{\mathbf{k}}=\dot{\phi}_\nu/\phi_\nu=\pi\Xi(\delta)/\partial_\omega\Xi(\mathscr{P})$ with the principal-value distribution $\mathscr{P}(\cdot)=\mathrm{p.v.}(1/\cdot)$, and $\Xi(f)=\sum_{\mu,j}\mathcal{A}_{\mu j}J^2_{0\mu\nu}\{\lambda^{+}_{\mu j0}f(\Omega^{+}_{\mu j0})-\lambda^{-}_{\mu j0}f(\Omega^{-}_{\mu j0})\}$, where f denotes a generic one-dimensional distribution. To lowest order in $\dot{\phi}_\nu$, the real part of (8) gives the dispersion relation $\epsilon(\mathbf{k},\omega_{\mathbf{k}})=0$, where $\epsilon(\mathbf{k},\omega_{\mathbf{k}})=-\chi_0\partial_r(\partial_r\phi_\nu/\chi_0)+(K-L\Xi(\mathscr{P}))\phi_\nu$. We now follow the analysis by introducting $\delta^2 v^{\pm}_{\mu j}$, the second-order perturbation in $v^{\pm}_{\mu j}$, such that $\langle\delta^2 v^{\pm}_{\mu j}\rangle_{\theta,z}\neq 0$. We then plug the decomposition $v^{\pm}_{\mu j}=v^{\pm}_{\mu j0}+\delta v^{\pm}_{\mu j}+\delta^2 v^{\pm}_{\mu j}$ into eqs. (5), (6) and use the WKB ansatz. The first term of the right-hand side of (6), the free-streaming part of the solution, will rapidly damp in (5) because of the phase mixing in the $\mathbf{k}$-integration on a time $\tau_{fs}\sim|\Delta(\omega_{\mathbf{k}}-\mathbf{k}\cdot\mathbf{V}^{\pm}_{\mu j0})|^{-1}$, provided $|\Delta(\omega_{\mathbf{k}} - \mathbf{k}\cdot\mathbf{V}^{\pm}_{\mu j0})|^{-1} \ll \gamma_k^{-1}, \tau_{dif}$. Dropping the free-streaming term because phase mixing leads to its decay on a time τ_{fs}, after (θ, z)-averaging and neglecting third-order terms in the perturbation we get

$$\partial_t v^{\pm}_{\mu j0} + \partial_t \langle \delta^2 v^{\pm}_{\mu j}\rangle_{\theta,z} + \frac{1}{r}\partial_r\left(\mathcal{J}^{\pm}_{\mu j}\right) = 0, \tag{9}$$

where

$$\mathcal{J}^{\pm}_{\mu j}(t) = r\int_0^t \sum_{\nu\neq\mathbf{0}} \lambda^{\pm}_{\mu j0}(t-s)\frac{k_\theta}{B}J^2_{0\mu\nu} \times \phi_\nu(t-s)\bar{\phi}_\nu(t)\exp\left(i\int_{t-s}^t \Omega^{\pm}_{\mu j0}(\tau)\mathrm{d}\tau\right)\mathrm{d}s. \tag{10}$$

Again by phase mixing the sum over the mode ν or $\mathbf{k}$-integration in expression (10) leads to its decay in s on time τ_{fs} which justifies the extension of the s-integration to infinity in eq. (10). Moreover if the ordering $\tau_{fs} \ll \gamma_{\mathbf{k}}^{-1}$ holds, then Taylor expansions of slowly time-evolving unknowns are justified. As a result, second-order terms in time can be neglected and assuming the ordering $\tau_{dif} \gg \gamma_{\mathbf{k}}^{-1}$, eq. (10) becomes

$$\mathcal{J}^{\pm}_{\mu j} = \sum_{\nu\in\mathbb{D}} \frac{rk_\theta}{B}\lambda^{\pm}_{\mu j0}\left(2\pi\delta(\Omega^{\pm}_{\mu j0}) - \partial_\omega\mathscr{P}(\Omega^{\pm}_{\mu j0})\frac{\mathrm{d}}{\mathrm{d}t}\right)|J_{0\mu\nu}\phi_\nu|^2, \tag{11}$$

where $\mathbb{D} = \{\nu = (m,n)\ m>0,\ n\neq 0\}$. We now substitute the expression (11) into (9) and associate respectively the first term of the right-hand side of (11) to $\partial_t v^{\pm}_{\mu j0}$ and the second term of the right-hand side of (11) to $\partial_t\langle\delta^2 v^{\pm}_{\mu j}\rangle_{\theta,z}$. As a result we obtain the set of nonlinear diffusion equations

$$\frac{\partial v^{\pm}_{\mu j0}}{\partial t} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\mathcal{D}^{\pm}_{\mu j}\frac{\partial v^{\pm}_{\mu j0}}{\partial r}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\Pi^{\pm}_{\mu j,r,\parallel}\right) \tag{12}$$

with resonant positive diffusion coefficients

$$\mathcal{D}^{\pm}_{\mu j} = 2\pi\rho_s c_s \sum_{\nu\in\mathbb{D}} Z_i^2\rho_s^2 k_\theta^2 \left|\frac{eJ_{0\mu\nu}\phi_\nu}{T_e}\right|^2 \Omega_i\delta(\omega_{\mathbf{k}} - \mathbf{k}\cdot\mathbf{V}^{\pm}_{\mu j0}),$$

and fluctuation-driven parallel Reynolds stresses

$$\Pi^{\pm}_{\mu j,r,\parallel} = 2\pi\sum_{\nu\in\mathbb{D}} Z_i^2 c_s^2\rho_s^2 k_\theta k_\parallel \left|\frac{eJ_{0\mu\nu}\phi_\nu}{T_e}\right|^2 \Omega_i\delta(\omega_{\mathbf{k}} - \mathbf{k}\cdot\mathbf{V}^{\pm}_{\mu j0}).$$

Equations (12) have the same physical sense as eq. (4) derived in [6], in the Vlasovian kinetic framework. The diffusion coefficient $\mathcal{D}^{\pm}_{\mu j}$ accounts for the familiar radial spatial scattering of each bag independently, by stochastic $E\times B$ drifts. The non-diffusive off-diagonal term $\Pi^{\pm}_{\mu j,r,\parallel}$ represents the correlated radial scattering and parallel acceleration which is related to the generation of a radially sheared parallel flow. Such off-diagonal term contributes to the turbulent momentum flux, producing possible inward convection by momentum transfer between resonant particles and waves. This radially sheared parallel flow can lead to new phenomena such as intrinsic rotation and momentum transport bifurcations whose interplay with internal transport barrier dynamics and confinement improvements give rise to intense interest [5,6]. Let us notice that here neither electric field shear nor toroidicity are necessarily required for off-diagonal momentum flux contribution. Using the resonance condition, $\omega_{\mathbf{k}} - \mathbf{k}\cdot\mathbf{V}^{\pm}_{\mu j0} = 0$, eqs. (12) can be recast into Fokker-Planck equations

$$\frac{\partial v^{\pm}_{\mu j0}}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\mathcal{F}^{\pm}_{\mu j}v^{\pm}_{\mu j0} + r\mathcal{D}^{\pm}_{\mu j}\frac{\partial v^{\pm}_{\mu j0}}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(rQ^{\pm}_{\mu j}\frac{\partial\phi_0}{\partial r}\right) \tag{13}$$

with friction coefficients

$$\mathcal{F}^{\pm}_{\mu j} = \sum_{\nu\in\mathbb{D}} \mathcal{G}_\nu\left(1+\mathcal{I}_\nu\right)\left|\frac{eJ_{0\mu\nu}\phi_\nu}{T_e}\right|^2 \Omega_i\delta(\omega_{\mathbf{k}} - \mathbf{k}\cdot\mathbf{V}^{\pm}_{\mu j0}),$$

and source coefficients

$$Q^{\pm}_{\mu j} = \sum_{\nu\in\mathbb{D}} \mathcal{G}_\nu\mathcal{I}_\nu\frac{k_\theta}{k_\parallel B}\left|\frac{eJ_{0\mu\nu}\phi_\nu}{T_e}\right|^2 \Omega_i\delta(\omega_{\mathbf{k}} - \mathbf{k}\cdot\mathbf{V}^{\pm}_{\mu j0}),$$

where $\mathcal{G}_\nu = -2\pi Z_i^2 c_s^2\rho_s^2 k_\parallel^2 k_\theta\omega_{\mathbf{k}}^{-1}$ and $\mathcal{I}_\nu = \omega_{\mathbf{k}}^{-1}k_\theta\partial_r\phi_0/B$. The bag equations (13) are coupled by the equation of the zonal flow

$$-\chi_0\partial_r\left(\frac{\partial_r\phi_0}{\chi_0}\right) + \kappa(1-\lambda)\phi_0 = L\left(\sum_{\substack{\mu\in\Lambda\\ j\leqslant\mathcal{N}}}\mathcal{A}_{\mu j}(v^{+}_{\mu j0} - v^{-}_{\mu j0}) - \frac{q_i}{2\pi\Omega_i}n_{i0}\right). \tag{14}$$

The existence of the source term due to $Q^{\pm}_j$ in (13) couples the bags dynamics with the zonal flow given self-consistently by eq. (14). Consequently the system (13), (14) is a reaction-diffusion system describing the weak-turbulence diffusion (direct energy cascade) which enters in competition with the back reaction driven by the nonlinear diffusion term (last term in (13)) on the mean —zonal— flow (inverse energy cascade). In such a plasma the back reaction of self-generated shear flow on pressure-gradient–driven turbulence is a key mechanism that governs the turbulent state and the transport, especially it can lead to the formation of transport barriers.

Futhermore our model seems, indeed, to have some universal properties for the reaction-diffusion process. In the one-bag case, with $v^+ = v^-$ (zero temperature limit) eqs. (13), (14) have the same structure as the Keller-Segel model [7], widely used in chemotaxis to describe the collective motion (diffusion and aggregation) of cells in multicellular organisms.

We can now integrate in time the equation on $\langle\delta^2 v^{\pm}_{\mu j}\rangle_{\theta,z}$ to retrieve conservation laws (density, momentum and energy) at the second order in the perturbation. An integration in time of the equation for the term $\langle\delta^2 v^{\pm}_{\mu j}\rangle_{\theta,z}$ gives

$$\langle\delta^2 v^{\pm}_{\mu j}\rangle_{\theta,z} = \frac{1}{r}\partial_r\sum_{\nu\in\mathbb{D}}\frac{rk_\theta}{B}\lambda^{\pm}_{\mu j0}|J_{0\mu\nu}\phi_\nu|^2\partial_{\omega_{\mathbf{k}}}\mathscr{P}(\Omega^{\pm}_{\mu j0}). \tag{15}$$

As we will see below, the term $\langle\delta^2 v_{\mu j}^{\pm}\rangle_{\theta,z}$ enables us to retrieve conservation laws (density, momentum and energy) at the second order in the perturbation. From eq. (13), after integrating over the cylinder we get easily the conservation of the total density

$$\frac{\mathrm{d}n_0}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(4\pi^2 L_z \frac{\Omega_i}{q_i}\sum_{\mu,j}\mathcal{A}_{\mu j}\int r\mathrm{d}r(v_{\mu j0}^{+} - v_{\mu j0}^{-})\right) = 0.$$

Let us now define $\mathscr{K}_{0\ell}$ the nonresonant particle momentum for $\ell = 2$ and energy for $\ell = 3$ at zero order as

$$\mathscr{K}_{0\ell} = 4\pi^2 B L_z \frac{1}{\ell}\int r\mathrm{d}r \sum_{\mu,j}\mathcal{A}_{\mu j}\left({v_{\mu j0}^{+}}^{\ell} - {v_{\mu j0}^{-}}^{\ell}\right),$$

and $\mathscr{K}_{2\ell}$ the resonant particle contribution to the wave momentum for $\ell = 1$ and to the wave energy for $\ell = 2$

$$\begin{aligned}\mathscr{K}_{2\ell} \;&=\; 4\pi^2 B L_z \int r\mathrm{d}r \sum_{\mu,j}\mathcal{A}_{\mu j}\\ &\times\left({v_{\mu j0}^{+}}^{\ell}\langle\delta^2 v_{\mu j}^{+}\rangle_{\theta,z} - {v_{\mu j0}^{-}}^{\ell}\langle\delta^2 v_{\mu j}^{-}\rangle_{\theta,z}\right),\end{aligned}$$

and finally the total wave energy $\mathscr{E}$,

$$\mathscr{E} = 4\pi^2 B L_z \int r\mathrm{d}r \sum_{\nu\in\mathbb{D}} 2\left(\frac{k_\theta}{Bk_\parallel}\right)^2 \frac{\partial_r^2\phi_0}{L}\phi_\nu\omega\partial_\omega\epsilon.$$

Using eqs. (13), (15) and the dispersion relation (8) after some algebra we get that $\dot{\mathscr{K}}_{02} + \dot{\mathscr{K}}_{21} = 0$ and $\dot{\mathscr{K}}_{03} + \dot{\mathscr{E}} = \dot{\mathscr{K}}_{03} + \dot{\mathscr{K}}_{22} + \dot{\mathscr{E}}_\phi = 0$ which means that the momentum and the energy, respectively, are conserved at second order in the perturbation. The term $\mathscr{E}_\phi = \mathscr{E} - \mathscr{K}_{22}$ represents the electrical potential energy.

Conclusion. – We have presented a new model, named the gyro-water-bag, to study gyrokinetic turbulent diffusion in nearly collisionless fusion plasmas. This model is obtained from a reduction of the six-dimensional phase space of the Vlasov equation through the existence of two underlying invariants, the magnetic moment being adiabatic and the "bag" being exact. In order to understand the nature of the transport, we have developed the weak-turbulence theory of the gyro-water-bag leading to nonlinear diffusion equations with fuctuation-driven parallel Reynolds stresses as source terms which are responsible for the generation of sheared parallel flow and momentum transport bifurcations which may impact transport barrier dynamics and confinement. The resonance condition allows to recast these equations into Fokker-Planck equations with nonlinear diffusion source terms on the mean flow (zonal flow), which lead to the set of coupled reaction-diffusion equations. This is the back reaction of the turbulent diffusion which can lead to the formation of transport barriers. Therefore we have derived a model which can describe the radial turbulent diffusion process and its interplay with the back reaction of sheared parallel flow and poloidal zonal flow. It is also important to keep in mind that the superposition of several bags allows to recover nonlinear Landau resonances of the phase-space flow by phase-mixing process of real frequencies which is reminiscent of the Van Kampen-Case [13,14] treatment of electronic plasma oscillations. A noteworthy fact is that our reaction-diffusion model has some universal properties for the reaction-diffusion process since the mathematical structure is the same as the famous Keller-Segel model [7], widely used in chemotaxis to describe the collective motion of cells in multicellular organisms. Through a quasilinear analysis we have derived semi-analytical transport coefficients which take into account nonlinear Landau resonances to predict the level of this turbulence. We have also shown that our weak-turbulence gyro-water-bag model preserves the classical conservation laws. In order to check the hypothesis of this quasilinear approach, comparisons with numerical resolution of the gyro-water-bag have been carried out and will be the scope of forthcoming papers.

REFERENCES

[1] DePackh D. C., *J. Electron. Control*, **13** (1962) 417.
[2] Feix M. R., Hohl F. and Staton L. D., in *Nonlinear effects in Plasmas*, edited by Kalmann G. and Feix M. R. (Gordon and Breach), 1969, pp. 3–21.
[3] Bertrand P. and Feix M. R., *Phys. Lett. A*, **28** (1968) 68.
[4] Morel P., Gravier E., Besse N., Klein R., Ghizzo A., Bertrand P., Garbet X., Ghendrih Ph., Grandgirard V. and Sarazin Y., *Phys. Plasmas*, **14** (2007) 112109.
[5] Garbet X., Sarazin Y., Ghendrih P., Benkadda S., Beyer P., Figarella C. and Voitsekhovitch I., *Phys. Plasmas*, **9** (2002) 3893.
[6] Diamond P. H., McDevitt C. J., Gürcan Ö. D., Hahm T. S. and Naulin V., *Phys. Plasmas*, **15** (2008) 012303.
[7] Keller E. F. and Segel L. A., *J. Theor. Biol.*, **26** (1970) 399.
[8] Frieman E. A. and Chen L., *Phys. Fluids*, **25** (1982) 502.
[9] Dubin D. H. E., Krommes J. A., Oberman C. and Lee W. W., *Phys. Fluids*, **26** (1983) 3524.
[10] Hahm T. S., *Phys. Fluids*, **31** (1988) 2670.
[11] Brizard A. J. and Hahm T. S., *Rev. Mod. Phys.*, **79** (2007) 421.
[12] Brizard A. J., *Phys. Rev. Lett.*, **84** (2000) 5768.
[13] Van Kampen N. G., *Physica*, **21** (1955) 949.
[14] Case K. M., *Ann. Phys. (N.Y.)*, **7** (1959) 349.
[15] Sagdeev R. Z. and Galeev A., *Nonlinear Plasma Theory* (Benjamin, New York) 1969.
[16] Vedenov A. A., Gordeev A. V. and Rudakov L. I., *Plasma Phys.*, **9** (1967) 719.
[17] Drummond W. E. and Pines P., *Nucl. Fusion*, **SUP3** (1962) 1049.
[18] Dupree T. H., *Phys. Fluids*, **9** (1966) 1773.