Chap. 1 - Differential Equations

EDX1

I - Introduction

Ordinary Differential Equation :

relation between a function $y(x)$ and its derivatives :

$$\boxed{F(x, y, y', \dots, y^{(n)}) = 0}$$

n : order of the equation.

ex: $y'' + 3y + \sin x = 0 \rightarrow$ order 2

$y''^3 + 3y + \sin x = 0 \rightarrow$ order 2, degree 3

* linear O. diff. eq. :

(1) $\boxed{a_0(x)\, y + a_1(x)\, y' + a_2(x)\, y'' + \dots + a_n(x)\, y^{(n)} = f(x)}$

$\uparrow$ rhs

homogeneous equation associated to (1)

$$a_0(x)\, y + \dots + a_n(x)\, y^{(n)} = 0$$

* Quasi - linear O. D. E :

linear in its highest derivative.

ex : $y'' + y^2 = \sin x$

II. First - Order equations

1) Example

$$y' + ky = e^{2x}, \quad k \text{ constant.}$$

Solution of the form $y = Y + y_0$

with Y sol. of the homogeneous eq. $Y' + kY = 0$

y_0 a particular solution

proof. y general sol of $y' + ky = e^{2x}$

y_0 particular integral $y_0' + ky_0 = e^{2x}$

by difference : $\underbrace{(y - y_0)'}_{Y'} + k\underbrace{(y - y_0)}_{Y} = 0$

$\Rightarrow y = Y + y_0$

- here $Y = c\, e^{-kx}$ c = constant.
- find y_0 : use of the variation of parameters / constants (technique discovered by Laplace). ⚠ linear eq. only!

Consider c variable and write $y_0(x) = C(x)\, e^{-kx}$

Then : $y_0' = C'(x)\, e^{-kx} - k\, C(x)\, e^{-kx}$

and making use of the ED:

$$C'(x)\, e^{-kx} - k\, C(x)\, e^{-kx} + k\, C(x)\, e^{-kx} = e^{2x}$$

$$\Rightarrow \quad C'(x) = e^{(k+2)x}$$

$$\Rightarrow \quad C(x) = \frac{1}{k+2}\, e^{(k+2)x}$$

Thus : $\boxed{y(x) = c\, e^{-kx} + \frac{1}{k+2}\, e^{2x}}$

Find c : use of initial / boundary conditions.

2) Separable equations

Equations of the form

$$A(x)\, dx + B(y)\, dy = 0$$

or

$$g(x) = f(y)\, y'$$

Solution is immediate by integration.

ex :

$$\frac{dy}{\sqrt{1-y^2}} + \frac{dx}{\sqrt{1-x^2}} = 0$$

$$\Rightarrow \quad \arcsin y = -\arcsin x + cte$$

$$\Rightarrow \quad \boxed{y = \sin(cte - \arcsin x)}$$

constant is found with initial cond.

3) Exact differential equations

Equations of the form :

$$A(x,y)\, dx + B(x,y)\, dy = 0$$

where the l.h.s. is a total differential i.e. : $A(x,y)\,dx + B(x,y)\,dy = dU(x,y)$

where $A = \dfrac{\partial U}{\partial x}$ and must verify $\dfrac{\partial A}{\partial y} = \dfrac{\partial B}{\partial x}$ $\left(= \dfrac{\partial^2 U}{\partial x \partial y}\right)$

$B = \dfrac{\partial U}{\partial y}$.

in that case : find U by integration of $\frac{\partial U}{\partial x} = A$

then derive the result w/r to y and identify to $B = \frac{\partial U}{\partial y}$.

then : $dU = 0 \Rightarrow U = cte$ gives the solution $y(x)$

Example

$\underbrace{(x+y)}_{A}\,dx + \underbrace{x}_{B}\,dy = 0$ (non separable)

$\dfrac{\partial A}{\partial y} = 1 = \dfrac{\partial B}{\partial x} \Rightarrow$ ok

take $A = x+y = \dfrac{\partial U}{\partial x} \Rightarrow U = \dfrac{x^2}{2} + xy + f(y)$

take $\dfrac{\partial U}{\partial y} = x + f'(y)$

identify to B : $x + f'(y) = x \Rightarrow f(y) = C_1$ (cte)

$\Rightarrow$ Solution is $U = \dfrac{x^2}{2} + xy + \cancel{cte} = C_2$ (cte)

$$\Rightarrow \boxed{y(x) = \frac{C}{x} - \frac{x}{2}}$$

* Integrating factor

If the equation is not exact, it is sometimes possible to find a function $\lambda(x,y)$ so that $\lambda(A\,dx + B\,dy)$ becomes exact.

$\lambda(x,y)$ is called an integrating factor.*

→ there is no general way to find λ, excepted if λ is a function of x or y alone.

(anyway the method is not always working and it is required that we verify that $\lambda(A\,dx + B\,dy)$ is an exact diff.

* Thermo: $\frac{1}{T}$ is the integrating factor of δQ to give $du = \frac{\delta Q}{T} = T\,ds - P\,dV$

→ imagine we search an integrating factor $\lambda(x)$.

then :

$$\boxed{\lambda(x) = \exp\left[\int \frac{1}{B} \left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right) dx\right]}$$

proof: we want $\lambda A \, dx + \lambda B \, dy$ to be an exact diff

$$\Rightarrow \frac{\partial}{\partial y}(\lambda A) = \frac{\partial}{\partial x}(\lambda B)$$

$$\Rightarrow \lambda \frac{\partial A}{\partial y} = \lambda'(x) B + \lambda \frac{\partial B}{\partial x}$$ divide by $\frac{1}{\lambda B}$

$$\Rightarrow \frac{1}{B}\frac{\partial A}{\partial y} = \frac{\lambda'}{\lambda} + \frac{1}{B}\frac{\partial B}{\partial x}$$

$$\Rightarrow \ln(\lambda) = \int \frac{1}{B}\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right)$$

(no integration constant necessary: useless)

→ if λ is to be a function of y:

$$\boxed{\lambda(y) = \exp\left[\int \frac{1}{A}\left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) dy\right]}$$

example:

$$xy' + (1+x)y = e^x$$

$$\Rightarrow \underbrace{\vphantom{\frac{1}{x}}}_{B} dy + \underbrace{\left[\left(\frac{1+x}{x}\right)y - \frac{e^x}{x}\right]}_{A} dx = 0$$

$$\frac{\partial B}{\partial x} = 0 \qquad \frac{\partial A}{\partial y} = 1 + \frac{1}{x} \Rightarrow \text{not an exact diff.}$$

try to find an integrating factor $\lambda(x)$:

$$\ln \lambda(x) = \int \left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right) dx = \int \left(1 + \frac{1}{x}\right) dx$$

$$= x + \ln(x) \Rightarrow \lambda(x) = x e^x.$$

we try :

$$\underbrace{x e^x}_{B} dy + \underbrace{\left[\left(e^x + x e^x\right) y - e^{2x}\right]}_{A} dx = 0$$

$$\frac{\partial B}{\partial x} = x e^x + e^x = \partial A / \partial y \quad \text{ok!}$$

$\Rightarrow$ find $U(x,y)$ so that

$$\frac{\partial U}{\partial x} = x e^x y + y e^x - e^{2x} \quad (1) \qquad \frac{\partial U}{\partial y} = x e^x \quad (2)$$

from (2) : $U = xy e^x + f(x)$

$$\Rightarrow \frac{\partial U}{\partial x} = y e^x + xy e^x + f'(x)$$

identify to (1) $\Rightarrow f'(x) = -e^{2x} \Rightarrow f(x) = -\frac{e^{2x}}{2}$

$$\Rightarrow U = xy e^x - \frac{e^{2x}}{2}$$

$U = cte \Rightarrow$ $\boxed{y = c\,\frac{e^{-x}}{x} + \frac{e^x}{2x}}$

⚠ $\lambda(x)$ must not be function of y
$\lambda(y)$ ——— x

ex : $x\,dx + (x+y)\,dy = 0$

search for $\lambda(x)$ gives : $\lambda = \frac{1}{x+y}$

search for $\lambda(y)$ gives : $\lambda = e^{y/x}$

$\Rightarrow$ the method fails

(and the integrating factor does not make the equation exact).

4) Change of variable

Idea : to make the equation exact or separable.

a) Equations of type $y' = f(ax + by + c)$

take $v = ax + by + c$ as new variable

ex : $y' = x + y \iff dy = (x+y)\,dx$.

take $v = x + y \Rightarrow dx = dv - dy$

$$\Rightarrow dy = v\,(dv - dy) \Rightarrow dy = \frac{v}{1+v}\,dv$$

separable.

integration by parts gives: $\int \frac{v}{1+v} = v - \ln(1+v) + 1$

$\Rightarrow \quad y = v - \ln(1+v) + cte$

$\Rightarrow \quad y = x + y - \ln(1+x+y) + cte$

$\Rightarrow \quad \ln(x+y+1) = x + cte \quad \Rightarrow \quad x+y+1 = cte\, e^{x}$

and : $\boxed{y = cte\, e^{x} - x - 1}$

NB: Eq. could have been solved by the classical way of § II.1 (homogeneous + var. of constant)

b) Equations with Homogeneous functions

⚠ not to be confused with the homogeneous eq. of type $F(y, y', \dots) = 0$ of § I.

→ a function $f(x, y, \dots)$ is homogeneous of degree r if $f(ax, ay, \dots) = a^{r} f(x, y, \dots)$

→ A ODE $A(x,y)\,dx + B(x,y)\,dy = 0$

is homogeneous if A and B are homogeneous of the same degree.

In that case apply the change $y = vx$ to make the equation separable.

proof: the eq. $A(x,y)\,dx + B(x,y)\,dy = 0$

can be written as

$$\frac{dy}{dx} = F(x,y) \quad \text{with} \quad F(ax, ay) = F(x,y)$$

$$\left(F = -\frac{A}{B}\right)$$

let $v = y/x$

$$\Rightarrow \frac{dy}{dx} = x\frac{dv}{dx} + v = F(x; vx) = F(1, v)$$

$$\Rightarrow \boxed{\frac{v'}{F(1,v) - v} = \frac{1}{x}} \quad \text{separable}$$

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ex: $\underbrace{y}_{A}\, dx + \underbrace{(2\sqrt{xy} - x)}_{B}\, dy = 0$

A and B homogeneous of degree 1

let $y = vx \Rightarrow dy = v\,dx + x\,dv$

$\Rightarrow vx\,dx + (2x\sqrt{v} - x)(v\,dx + x\,dv) = 0$

$\Rightarrow dx\,(2v^{3/2}) + (2\sqrt{v} - 1)\,x\,dv = 0$

$\Rightarrow dv\,\underbrace{\dfrac{(2\sqrt{v}-1)}{2v^{3/2}}} = -\dfrac{dx}{x}$

ou prendre directement :

$\dfrac{dx}{x} = \dfrac{dv}{F(1,v) - v}$

primitive:

$\ln(v) + \dfrac{1}{\sqrt{v}}$

$\Rightarrow \ln(v) + \dfrac{1}{\sqrt{v}} = -\ln(x)$

d'où $\boxed{\ln(y) + \sqrt{\dfrac{x}{y}} = cte}$

in that case we have no explicit form for $y(x)$.

* Case where y is of dimension x^m (isobaric)

$[A\,dx + B\,dy] = [A][x] + [B][x]^m$

$\Rightarrow [A] = [B]^{m-1}$

$\Rightarrow$ if $A(ax, a^m y) = a^r A(x,y)$. (degree r)

$B(ax, a^m y) = a^{r-m+1} B(x,y)$

In that case try $y = v x^m$ (isobaric equation)

example: $xy^2(3y\,dx + x\,dy) - (2y\,dx - x\,dy) = 0.$

$\begin{cases} A(x,y) = (3xy^3 - 2y) & \text{ok for } m = -\frac{1}{2} \\ B(x,y) = (x^2y^2 - x) & \text{ok } m = -\frac{1}{2} \end{cases}$

$\begin{cases} A(ax, a^{-1/2}y) = a^{-1/2} A(x,y) \Rightarrow r = -1/2 \\ B(ax, a^{-1/2}y) = a\,B(x,y) \Rightarrow \text{ok because } r - m + 1 = 1 \end{cases}$

change of variable : $y = v\, x^{-1/2}$

or, better $y^2 = u\, x^{-1}$ (with $u = v^2$)

The eq. becomes:

$$(3u-2)\, y\, du + 5u(1-u)\, dy = 0 \quad \text{separable}$$

Solution : $y(u) = \text{cte.}\ u^{2/5}(u-1)^{1/5}$

c) fractions rationnelles

d) Bernoulli equation

$$\boxed{y' + f(x)\, y = g(x)\, y^n}$$

Not linear! quasi-linear.

try $v = y^{1-n}$:

→ first we divide the ED by y^n (providing $y \neq 0$)

$y = 0$ is a singular sol.

$$\frac{y'}{y^n} + f(x)\, \underbrace{y^{1-n}}_{v} = g(x)$$

$$v = y^{1-n} \Rightarrow v' = (1-n)\, y'\, y^{-n}$$

$$\Rightarrow \frac{y'}{y^n} = \frac{v'}{1-n}$$

$$\Rightarrow \boxed{\frac{v'}{1-n} + f(x)\, v = g(x)}$$

⇒ the equation is now linear.

Ex:

$$y' + \underbrace{\frac{y}{x}}_{f(x)} = \underbrace{2x^3}_{g(x)}\, y^4.$$

try $v = y^{-3}$ the equation becomes

$$\frac{v'}{-3} + \frac{v}{x} = 2x^3 \Rightarrow v' - \frac{3v}{x} = -6x^3$$

integrate the homogeneous eq. gives $V = \text{cte}\ x^3$

variation of constant gives $v_0 = -6x^4$

the solution is : $\boxed{v = \text{cte}\ x^3 - 6x^4}$

d) Rational functions

$$\boxed{y' = \frac{ax + by + c}{ex + fy + g}}$$

try: $\begin{cases} x = X + \alpha \\ y = Y + \beta \end{cases}$ with α, β solutions of the system $\begin{cases} a\alpha + b\beta + c = 0 \\ e\alpha + f\beta + c = 0 \end{cases}$

then: $$\frac{dY}{dX} = \frac{aX + bY}{eX + fY} = F(X, Y)$$ homogeneous of degree 0

$\Downarrow$

goto § 2)

e) Clairaut Equation

$$\boxed{y - xy' = f(y')}$$

NOT linear

solved by differentiating: one obtains:

$$y'' \left[f'(y') + x \right] = 0$$

- general solution: $y'' = 0$

$\Rightarrow y = ax + b$

substituting into the Clairaut eq gives $b = f(a)$

$\Rightarrow$ general solution is:

$$y = ax + f(a)$$

- Singular solution: $f'(y') + x = 0$.

gives $y' = \arg f'(-x)$

where $\arg f'$ is the inverse function of f' ex: arccos for sin

and substituting into the D.E.

we obtain a relation between y and x with no integration constant.

$\Rightarrow$ there is no unicity of the solution (typical of non lin. ED) ex: $y = xy' + (y')^2$ $\quad f(y') = y'^2$

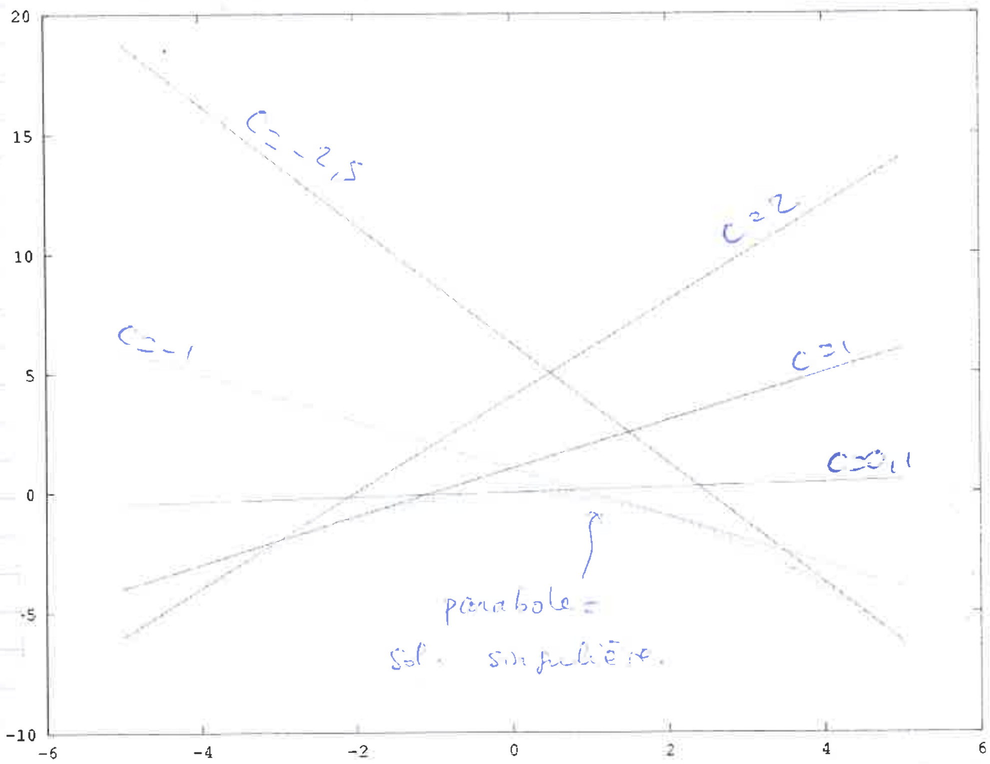
$\Rightarrow y''[2y' + x] = 0$

general solution: $\boxed{y = Cx + C^2}$

singular solution

$y' = -\frac{x}{2} \Rightarrow \boxed{y = -\frac{x^2}{2} + \left(\frac{x}{2}\right)^2 = -\frac{x^2}{4}}$

graphe des solutions:
des droites inclinées et une parabole.
chaque droite tangente la parabole en un point.



applications à l'optique:

pour une surface d'onde avec des aberrations, la trajectoire des rayons et la caustique sont les sol^n générale et singulière d'une éq. de Clairaut.

Singular solution often appear in non-linear dif. eq., when the solution at one point or one line $y(x)$ is not unique.

f) High degree first order equations

the D.E. is a polynom in y':

$$y'^n + a_{n-1}(x,y)\, y'^{(n-1)} + \dots + a_0(x,y) = 0.$$

* case where the D.E. is factorizable (solvable for p)

$$\Rightarrow \quad (p - F_1(x,y))\,(p - F_2(x,y))\dots = 0$$

with $p = y'$

every expression $p - F_i(x,y)$ is a 1^{st} degree eq. of solution given by $G_i(x,y) = 0$ (implicit form)

⚠ only one arbitrary constant (1st order)

Ex:

$$xy\,y'^2 + y'(x^2 + xy + y^2) + x^2 + xy = 0$$

se factorise en

$$(px + x + y)(x + y p) = 0$$
$$\quad (1) \qquad\qquad (2)$$

sol. of (1): $y'x + x + y = 0$

sol: $y(x) = -\frac{x}{2} + \frac{c}{x}$

sol of (2): $x + y\frac{dy}{dx} = 0$

$\hookrightarrow \quad y\,dy = -x\,dx$

$\Rightarrow \quad \frac{y^2}{2} = -\frac{x^2}{2} + cte$

$\Rightarrow \quad y^2 = -x^2 + cte$

$$\Rightarrow \begin{cases} G_1(x,y) = y + \frac{x}{2} - \frac{c}{x} \\ G_2(x,y) = y^2 + x^2 - c \end{cases} \quad \text{or}$$

a product is 0 if ~~either~~ one of its factors are 0 then the above syst. can be written as

$$G_1 \times G_2 = 0$$

$$\Rightarrow \quad \left[\left(y + \frac{x}{2} - \frac{c}{x}\right)\left(y^2 + x^2 - c\right) = 0\right]$$

with C the undetermined constant.

* D.E. solvable for x

i.e. eq. that can be written in the form:

$$x = F(y, y')$$

ex: $6y^2p^2 + 3px - y = 0 \qquad (p = y')$

can be written as:

$$3x = \frac{y}{p} - 6y^2 p$$

→ differentiate /y

$$\underbrace{3\frac{dx}{dy}}_{\frac{3}{p}} = \frac{1}{p} - \frac{y}{p^2}p'(y) - 6y^2\frac{dp}{dy} - 12yp$$

$$\Rightarrow \underbrace{(1 + 6yp^2)}_{(1)}\underbrace{\left(2p + y\frac{dp}{dy}\right)}_{(2)} = 0$$

So: (1) = 0 or (2) = 0

(2) = 0 gives $py^2 = cte$ and one solution to the ED

(1) = 0 gives $p^2 = -\frac{1}{6y}$ and a solution with no constant (singular sol°)

* DE solvable for y (ex: Clairaut)

similar to the DE solvable for x.

One finds generally a regular and a singular sol°

ex: $p^2x + 2xp - y = 0$

can be factorized into

$$\underbrace{(p+1)}_{}\underbrace{\left(p + 2x\frac{dp}{dx}\right)}_{(2)}$$

gives the singular solution $x + y = 0$.

(2) → $xp^2 = cte$

→ general sol^n: $(y - c)^2 = 4cx$

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III – Equations of higher ~~degree~~ order

1) Linear eq. with constant coefficients.

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_0 y = f(x).$$

the eq. is linear : solution is the sum of

→ the general solution $Y(x)$ to the homogeneous eq. (complementary function)

→ a particular integral $y_0(x)$

a) Complementary function

in this paragraph we discuss the solution $Y(x)$ of the homogeneous equation:

$$a_n Y^{(n)} + a_{n-1} Y^{(n-1)} + \dots + a_0 Y = 0 \qquad (1)$$

This equation admits a set of n independants solutions $(y_1, y_2, \dots y_n)$. The general solution is the linear combinaison

basis of the space of solutions

$$Y(x) = C_1 y_1(x) + C_2 y_2(x) + \dots$$

where the $C_1, \dots C_n$ are determined by the boundary values.

the characteristic equation associated to (1) is

$$a_n \lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_0 \lambda = 0$$ → n roots λ_p (real or complex)

Solutions generally have the form:

$$y_p(x) = e^{\lambda_p x}$$

excepted if λ_p is a multiple root : on that case we have less than n different solutions and need to find the others.

ex: imagine $\lambda_1 = \lambda_2$ (double root λ_1)

⇒ only $(n-1)$ independent sol^n $y_p = e^{\lambda_p x}$

if $\lambda_1 \neq \lambda_2$: $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$ are solutions,

so the linear combinaison $\dfrac{e^{\lambda_1 x} - e^{\lambda_2 x}}{\lambda_1 - \lambda_2}$ is also solution.

When $\lambda_1 \to \lambda_2$ this solution becomes

$$\lim_{\lambda_1 \to \lambda_2} \frac{e^{\lambda_1 x} - e^{\lambda_2 x}}{\lambda_1 - \lambda_2} = \frac{d}{d\lambda}\left(e^{\lambda x}\right)\Big|_{\lambda=\lambda_1} = \underbrace{x e^{\lambda_1 x}}_{\text{missing solution.}}$$

So the basis of the space of solutions is

$$\left(e^{\lambda_1 x},\ x e^{\lambda_1 x},\ e^{\lambda_2 x}, \dots\ e^{\lambda_{n-1} x}\right)$$

if a root is ~~do~~ triple $(\lambda_1 = \lambda_2 = \lambda_3)$ the solutions are

$$e^{\lambda_1 x},\ x e^{\lambda_1 x},\ x^2 e^{\lambda_1 x}, \dots$$

b) Particular integral

* lucky guess: (undetermined coefficients)

Sometimes a form for the particular integral can be tried:

→ if the 2^{nd} member (rhs) $f(x)$ is a polynom try a polynom for y_0.

→ if the 2^{nd} member is an exponential e^{mx} $(m \in \mathbb{C})$

try $y_0 = A e^{mx}$. If m is a solution of the characteristic equation, try a combination

$$A e^{mx} + B x e^{mx}$$

if f is polynom × exponential ⇓ try the same form.

→ For trigono. functions, try to solve on the complex plane

example:

$$y'' + 3y' + 2y = e^x$$

eq. characteristics: $\lambda^2 + 3\lambda + 2 = 0 \Rightarrow \lambda_1 = -2,\ \lambda_2 = -1$

$\Rightarrow$ ~~general~~ solution to the H.E. is

$$y = C_1 e^{-2x} + C_2 e^{-x}$$

particular integral: rhs is $e^x = e^{mx}$ with $m = 1$

$m = 1$ not solution of the chara. eq so we try

$$y_0(x) = A e^x$$

gives $A = \frac{1}{6}$

So the solution is $y(x) = C_1 e^{-2x} + C_2 e^{-x} + \frac{1}{6} e^x$

→ if the rhs is now e^{-x} :

-1 is solution of the char. eq.

$\Rightarrow$ try $y_0(x) = A e^{-x} + B x e^{-x}$ → inutile car déjà dans l'EH

put y_0 into the ED gives:

$$B e^{-x} = e^{-x} \Rightarrow B = 1, \text{ nothing for } A$$

⇓

any value is ok, take $A = 0$

$$\Rightarrow \underline{y_0(x) = x e^{-x}}$$

* **Variation of constants.**

the method works also for DE with non constant coefficients.

→ case of a 2nd order eq.

let $Y(x) = C_1 y_1(x) + C_2 y_2(x)$ the solution of the homog. eq.

We search a particular integral which verifies both of the following conditions:

$$\begin{cases} y_0(x) = C_1(x) y_1(x) + C_2(x) y_2(x) & (1) \\ y_0'(x) = C_1(x) y_1'(x) + C_2(x) y_2'(x) & (2) \end{cases}$$

{ for a 3rd order ED we would add the condition
$y_0''(x) = C_1(x) y_1''(x) + C_2(x) y_2''(x)$ etc... }

differentiate (1) $\Rightarrow$ $y_0' = C_1 y_1' + C_2 y_2' + C_1' y_1 + C_2' y_2$

combine with (2) $\Rightarrow$ $y_0' = C_1 y_1' + C_2 y_2'$

give the equation $\Rightarrow$ $\boxed{C_1'(x)\, y_1(x) + C_2'(x)\, y_2(x) = 0}$

We need another eq. to find both C_1' and C_2' : use the ED and put y_0 inside.

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example: $y'' + 3y' + 2y = \left(\frac{1}{x} - \frac{1}{x^2}\right) e^{-x}$

Solution of homog. eq: see above example.

$$Y(x) = c_1 e^{-2x} + c_2 e^{-x}$$

$$\Rightarrow \begin{cases} 2y_0 = 2c_1(x) e^{-2x} + 2c_2(x) e^{-x} \\ 3y_0' = \left(-2c_1 e^{-2x} - c_2 e^{-x}\right) \times 3 \\ y_0'' = 4c_1 e^{-2x} - 2c_1' e^{-2x} - c_2' e^{-x} + c_2 e^{-x} \end{cases}$$

→ using the ED, we have:

$$e^{-2x}\left(4c_1 - 2c_1' - 6c_1 + 2c_1\right) + e^{-x}\left(2c_2 - 3c_2 - c_2' + c_2\right) = \left(\frac{1}{x} - \frac{1}{x^2}\right)e^{-x}$$

$$\Rightarrow \quad \underline{-2c_1' e^{-x} - c_2' = \left(\frac{1}{x} - \frac{1}{x^2}\right)} \qquad \text{first eq.}$$

→ 2nd equation:

$$c_1' y_1 + c_2' y_2 = 0$$

$$\Rightarrow \quad c_1' e^{-2x} + c_2' e^{-x} = 0$$

$$\Rightarrow \quad \underline{c_1' e^{-x} + c_2' = 0} \qquad 2^{nd} \text{ eq.}$$

eliminate c_2' → $c_2' = -c_1' e^{-x}$

$$\Rightarrow \quad -c_1' e^{-x} = \left(\frac{1}{x} - \frac{1}{x^2}\right)$$

$$\Rightarrow \quad c_1' = \left(\frac{1}{x^2} - \frac{1}{x}\right) e^{x}$$

de la forme: $\underbrace{e^x}_{u'}\underbrace{\left(-\frac{1}{x}\right)}_{v} + \underbrace{e^x}_{u}\underbrace{\left(\frac{1}{x^2}\right)}_{v'}$

d'où $\boxed{c_1 = uv = -\frac{e^x}{x} \ (+\,cte)}$

$$\boxed{c_2' = -c_1' e^{-x} = \frac{1}{x} - \frac{1}{x^2} \Rightarrow c_2 = \ln(x) + \frac{1}{x} \ (+\,cte)}$$

we may not use the integration consts. since one particular solution is seeked.

$\Rightarrow$ the particular integral is:

$$\boxed{y_0 = C_1 e^{-2x} + C_2 e^{-x} = \ln(x) e^{-x}}$$

2) The Wronskian.

a n-th order homogeneous linear DE has a basis of n independent solutions $(y_1, \dots, y_n)$.

the solutions are independent if y_p cannot be wrote as a linear combination of the others, $\forall p$.

i.e: $\sum_i \alpha_i y_i \neq 0$ $(\alpha_i \neq 0)$.

$\hookrightarrow$ analog to the vector space (x, y):

$\vec{e}_1$ and $\vec{e}_2$ are 2 vectors linearly indep

if $\vec{e}_1$ not $/\!/$ to $\vec{e}_2$ $\Rightarrow$ $\vec{e}_1$ is not $\propto \vec{e}_2$.

in that case every vector $\vec{v}$ of the plane can be written in the form $\alpha \vec{e}_1 + \beta \vec{e}_2$ even if $\vec{e}_1$ and $\vec{e}_2$ are not orthogonal.

to check the independence of the functions $(y_1, \dots y_n)$ we compute the wronskian.

$$\boxed{W(y_1, \dots y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & & \vdots \\ y_1'' & & \ddots & \vdots \\ \vdots & & & \\ y_1^{(n)} & \cdots & & y_n^{(n-1)} \end{vmatrix}}$$

the functions are independent $\Longleftrightarrow$ the wronskian is $\neq 0$.

* case of the 2^{nd} order.

y_1, y_2 solutions of a linear ED.

$$W = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x) y_2'(x) - y_1'(x) \cdot y_2(x).$$

if y_2 and y_1 not indep., say $y_2 = \alpha y_1$, it is easy to see that $W = 0$

and if $W = 0 \Rightarrow y_1' y_2 = y_2 y_1'$

$$\Rightarrow \frac{y_1'}{y_1} = \frac{y_2'}{y_2} \Rightarrow \ln y_1 = \ln y_2 + cte$$

$$\Rightarrow y_1 = cte\, y_2 .$$

$\Rightarrow$ $W = 0 \iff (y_1, y_2)$ are independant.

* differential eq. of the wronskian: (order 2)

y_1 and y_2 verify $ay'' + by' + cy = 0$

$\hookrightarrow y'' = -\frac{b}{a} y' - \frac{c}{a} y$

$W = y_1 y_2' - y_2 y_1'$

$$\Rightarrow W' = \cancel{y_1' y_2'} + y_1 y_2'' - \cancel{y_2' y_1'} - y_2 y_1''$$

$$= y_1\left(-\frac{b}{a} y_2' - \cancel{\frac{c}{a} y_2}\right) - y_2\left(-\frac{b}{a} y_1' - \cancel{\frac{c}{a} y_1}\right)$$

$$= -\frac{b}{a}\left(y_1 y_2' - y_2 y_1'\right)$$

$$\Rightarrow \boxed{W' = -\frac{b(x)}{a(x)} W(x)}$$

D.E. order 1, separable

the solution is:

$$W(x) = W_0\, e^{-\int \frac{b(x)}{a(x)} dx}$$

W_0: constant.

Théorème de Liouville

$\Rightarrow$ Application: find another solution y_2 when y_1 is known, without solving directly the D.E.

ex:

$$x^2 y'' - xy' + y = 0$$

$y_1(x) = x$ is a trivial solution,

$\Rightarrow$ y_2 can be found with the wronskian.

$$\begin{cases} b(x) = -x \\ a(x) = x^2 \end{cases} \Rightarrow W = W_0 \exp - \int \frac{b(x)}{a(x)} dx = W_0 \exp + \int \frac{dx}{x}$$

$$[\, W = W_0 x$$

and with the definition of W we have:

$$W_0 x = y_1 y_2' - y_2 y_1' = x y_2' - y_2$$

easy to solve : $y_2' - \frac{y_2}{x} = W_0$

homog. solution : $Y_2 = C.x$

partic. integral : $y_0 = W_0 x \ln(x)$

$$\Rightarrow \boxed{y_2 = Cx + W_0 x \ln(x)}$$

3) Reduction of Order

(See next page)

4) Some particular cases.

a) Euler's D.E.

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \dots + a_0 y = 0$$

{the Eq is linear ⇒ if a rhs is $f(x)$ use a particular integral.}

Solutions are of the form $y_p(x) = x^\alpha$

where α is a root of ~~the~~ polynom (put y_α into the DE).

→ if p is a double root : add the solution $x^\alpha \ln(x)$

→ for multiple roots, the missing solutions are of the form $x^\alpha \ln(x)$, $x^\alpha \ln(x)^2$, ---

b) Equation $y'' = f(y)$

the order can be reduced by multiplying by y':

$$y' y'' = y' f(y) \Rightarrow \boxed{\frac{1}{2}(y')^2 = F(y)}$$

where $F(y)$ is a primitive of $f(y)$

c) Equation $y'' = f(y, y')$

Introduce the new function v: $y' = v(y)$

y' is a fn of y since x is not present

$\Rightarrow y'' = y' v'(y) = v'. v$

↳ analogue à ce qu'on fait pour RK avec un système : $v = y'$; $y' = \dots$

and plugging into the D.E:

$$\boxed{v' = \frac{f(y, v)}{v}} \rightarrow 1^{st} \text{ order D.E.}$$

ex: $y'' = \underbrace{(y')^3 y^2}_{f(y,y')}$.

change. $v = y'$

$$\Rightarrow v' = v^2 y^2 y' \Rightarrow \frac{v'}{v^2} = y^2 y' \Rightarrow -\frac{1}{v} = \frac{y^3}{3} + c^{te}$$

$$\Rightarrow v = -\frac{3}{y^3 + c_1}$$

$$\Rightarrow y' = v = \frac{-3}{y^3 + c_1}$$ DE. 1st order

$$\Rightarrow dy\,(y^3 + c_1) = -3\,dx$$

$$\Rightarrow \boxed{\frac{y^4}{4} + c_1 y = -3x + c_2}$$

3) Reduction of Order.

→ Alternative method to find y_2 if we know y_1.

Suppose the ED.

$$a(x)\,y'' + b(x)\,y' + y\,c(x) = 0$$

with $y_1(x)$ a particular integral that we know

⇒ try to find $y_2(x)$ of the form:

exercice: show that the term in v vanishes in the eq of v. ← $\boxed{y_2(x) = v(x) \cdot y_1(x)}$

ex: $2x^2 y'' + x y' - 3y = 0$ with $y_1(x) = \frac{1}{x}$

essayer la m̄ éq que le § Wronskien
$x^2y'' - xy' + y = 0$
avec $y_1 = x$.

let $y_2 = \frac{v(x)}{x} \Rightarrow y_2' = -\frac{v}{x^2} + \frac{v'}{x}$

$$\Rightarrow y_2'' = \frac{2v}{x^3} - \frac{2v'}{x^2} + \frac{v''}{x}$$

plug onto the ED and rearrange terms:

$$2x\,v'' - 3v' = 0 \qquad \text{take } U = v'$$

$$\Rightarrow 2xU' - 3U = 0 \qquad \text{Sol: } U = k \cdot x^{3/2} \Rightarrow v = k\,\frac{2}{5}\,x^{5/2}$$

$$\Rightarrow y_2 = c^{te} \cdot x^{3/2} \Rightarrow \boxed{y(x) = \frac{c_1}{x} + c_2 x^{3/2}}$$

→ we can also solve it by the Wronskian, which gives the same result:

$$W(x) = \frac{W_0}{\sqrt{x}} \Rightarrow y_2' + \frac{y_2}{x} = W_0 \sqrt{x}$$

homog. eq : $y_2 = \frac{c}{x}$

part. int. : $\frac{2}{5} W_0 x^{3/2}$

$$\Rightarrow y_2 = \frac{c}{x} + \frac{2}{5} W_0 x^{3/2}$$

↑ note that y_2 is different from the previous obtained by red. of order but $\frac{c}{x}$ is $\propto y_1$

$$\Rightarrow \boxed{y(x) = c_1 y_1 + c_2 y_2 = \frac{cte_1}{x} + cte_2\, x^{3/2}}$$

* Eliminating the term in y' instead of y:

$$a y'' + b y' + c y = 0 \qquad a, b, c \text{ functions of } x.$$

use the transformation : $y(x) = v(x)\, p(x)$

with $p(x) = \exp\left[-\frac{1}{2} \int \frac{b(x)}{a(x)}\, dx \right]$

$\Rightarrow p(x) = \sqrt{W(x)}$ wronskian.

Ex: Bessel eq:

$$x^2 y'' + x y' + (x^2 - n^2) y = 0$$

take $p(x) = \frac{1}{\sqrt{x}}$ $\Rightarrow v =$

⇒ the ED. becomes:

$$u'' + \left(1 - \frac{n^2 - \frac{1}{4}}{x^2}\right) u = 0$$

which is convenient to find the behavior at $x \to \infty$ where the eq simplifies in

$$u'' + u = 0 \Rightarrow \text{sine / cosine functions.}$$

$$\Rightarrow y \propto \frac{\text{sine / cosine}}{\sqrt{x}}$$

IV Power Series solutions.

1) Examples

Consider the equation $y' = y$; $y(0) = 1$

Solution is trivial, but we want a series of the form

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$\Rightarrow$ the problem is to calculate the c_n.

$$y' = \sum_{n=0}^{\infty} c_n \cdot n x^{n-1} = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$= \sum_{n=0}^{\infty} c_{n+1} (n+1) x^n$$

equalling y' to y and identify the coeffs. of same power gives:

$$c_n = (n+1) c_{n+1}$$

or, $\boxed{c_n = \frac{1}{n} c_{n-1}}$ = recurrence relation between the c_n.

$$= \frac{1}{n(n-1)} c_{n-2} = \frac{1}{n(n-1)(n-2)} c_{n-3}$$

$$= \frac{1}{n\cdot(n-1)\cdot\ldots 2 \cdot 1} c_0$$

$$\Rightarrow c_n = \frac{c_0}{n!}$$

c_0 derived from the boundary condition $y(0) = 1$. $\Rightarrow c_0 = 1$

giving $c_n = \frac{1}{n!}$ and

$$\boxed{y(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

But: if we try : $y' = -y^2$ with $y(1) = 1$

and search a form $y = \sum c_n x^n$ it will fail

since the solution is $y = \frac{1}{x}$ and diverges at $x = 0$

$\Rightarrow$ take care of convergence of the series !

* second example (as an exercice)

2nd order equation:

$$y'' - 2xy' + y = 0$$ linear & homogeneous

keep in mind $\Rightarrow y = a_1 y_1(x) + a_2 y_2(x)$

$$y = \sum_{n=0}^{\infty} c_n x^n \Rightarrow y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$$ (derivative of c_0 is 0 $\Rightarrow$ no term $n=0$)

$$y'' = \sum_{n=2}^{\infty} n(n-1) x^{n-2} c_n = \sum_{n=0}^{\infty} (n+1)(n+2) c_{n+2} x^n$$

$n := n+2$ (index shift)

plug into DE:

$$\sum_{n=0}^{\infty} (n+1)(n+2) c_{n+2} x^n - 2x \sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$$

⚠ convergence

$$\sum_{n=0}^{\infty} (n+1)(n+2) c_{n+2} x^n - \sum_{n=1}^{\infty} 2n c_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$

we can add the null term $n=0$ to the series

$$\Rightarrow \sum_{n=0}^{\infty} \left[(n+1)(n+2) c_{n+2} - (2n-1) c_n\right] x^n = 0 \quad \forall x$$

$\Rightarrow$ all terms vanish

$\Rightarrow$ recurrence relation:

$$\boxed{c_{n+2} = \frac{2n-1}{(n+1)(n+2)} c_n}$$

2 families of c_n's: odd and even.

→ given c_0 we obtain the even ones:

$$c_2 = -\frac{1}{2} c_0 \; ; \; c_4 = -\frac{3}{4!} c_0 \; ; \; c_6 = -\frac{1 \times 3 \times 7}{6!} c_0 \; ; \; c_8 = -\frac{3.5.7.11}{8!} c_0$$

$2n-5$

$$\Rightarrow \boxed{c_{2n} = -\frac{\prod_{k=2}^{n} (4k-5)}{(2n)!} c_0}$$

ED 24

→ given c_1 we obtain the odd ones.

$$c_3 = \frac{1}{3!} c_1 \; ; \; c_5 = \frac{1 \cdot 5}{5!} c_1 \; ; \; c_7 = \frac{1 \cdot 5 \cdot 9}{7!} c_1 \; ; \; c_9 = \frac{1 \cdot 5 \cdot 9 \cdot 13}{9!} c_1$$

$$\Rightarrow \boxed{c_{2n+1} = \frac{1}{(2n+1)!} \prod_{k=1}^{n} (4k-3) \cdot c_1}$$

d'où : name $\begin{cases} y_2 \\ y_1 \end{cases}$ the series with $\begin{cases} \text{odd} \\ \text{even} \end{cases}$ coefs :

⚠ providing y_1, y_2 converges

$$\begin{cases} y_1(x) = x + \dfrac{x^3}{6} + \displaystyle\sum_{n=2}^{\infty} \dfrac{5 \times 4 \times 9 \times \cdots \times (4n-3)}{(2n+1)!} x^{2n+1} \\ y_2(x) = 1 - \dfrac{x^2}{2} + \displaystyle\sum_{n=2}^{\infty} \dfrac{3 \cdot 7 \cdot 11 \cdots (4n-5)}{(2n)!} x^{2n} \end{cases}$$

we obtain $\left[y(x) = c_1 y_1 + c_0 y_2 \right.$

→ interesting if we want just an approximation at the origin :

$$y(x) \simeq c_0 + c_1 x - \frac{c_0}{2} x^2 + \cdots$$

→ if we want, for example (symetry argument) an odd function for y $\Rightarrow$ $c_0 = 0$

2) Solution near regular singular points.

a) Regular singular points

consider the equation : $a(x) y'' + b(x) y' + c(x) y = 0$

when a, b, c are analytic functions everywhere

(or in the whole interval $I \subset \mathbb{R}$ where we want to compute the solution).

define : $P(x) = \dfrac{b}{a}$; $q(x) = \dfrac{c}{a}$

to put in standart form :

$$y'' + P(x) y' + q(x) y = 0$$

→ a point x_0 is a singular point if $a(x_0)=0$
≡ pole of $p(x)$ ~~and~~ $q(x)$

→ x_0 is a regular singular point if

$$\begin{cases}(x-x_0)\,p(x) \\ (x-x_0)^2\,q(x)\end{cases}$$ are both analytic at x_0

i.e. x_0 is a $\begin{cases}\text{simple pole of } p \\ \text{double or simple pole of } q.\end{cases}$

⇒ series solution will fail to converge at x_0.
and near x_0, a lot of terms will be required
to approximate the function.

b) Frobenius method

idea: generalisation of the expansion series method
to give series solution up to the singular point.

⚠ works only for regular singular points.

→ In the following we assume $x_0=0$ the singular points.
(make a variable change if not).

We seek solutions of the form: $\boxed{y = x^r \sum_{n=0}^{\infty} c_n x^n}$ with $c_0 \neq 0$

Where r can be any real, positive or negative
and is determined by pluging y into the D.E.

★ example.

$$4x\,y'' + 2y' + y = 0 \quad (1)$$

standart form: $y'' + \frac{1}{2x} y' + \frac{1}{4x} y = 0$

⇒ 0 is a simple pole of $p(x)$ and $q(x)$

we try $y = x^r \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+r}$

$$\Rightarrow \quad y' = \sum_{n=0}^{\infty} c_n (n+r)\, x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} c_n (n+r)(n+r-1)\, x^{n+r-2}$$

plug onto the DE:

$$4\sum_0^\infty c_n(n+r)(n+r-1)x^{n+r-1} + 2\sum_0^\infty c_n(n+r)x^{n+r-1} + \sum_0^\infty c_n x^{n+r} = 0$$

we shift index $p=n+1 \Rightarrow n=p-1$

$$\sum_{p=1}^\infty c_{p-1}x^{p+r-1} \quad \text{let } c_{-1}=0$$

$$\sum_{p=0}^\infty c_{p-1}x^{p+r-1}$$

$$\Rightarrow \sum_{n=0}^\infty x^{n+r-1}\left\{4c_n(n+r)(n+r-1) + 2c_n(n+r) + c_{n-1}\right\} = 0$$

vanishes for all x if the $\{\}$ is 0

- lowest term: $n=0$

$$4c_0 r(r-1) + 2c_0 r = 0$$

indicial equation (indicielle en français)

gives allowed values of r.

here: $r_1 = 0$ and $r_2 = \frac{1}{2}$ → 2 series solution?

- recurrence relation for each r:

  × case $r_1 = 0$

$$\{\} = 4c_n n(n-1) + 2n c_n + c_{n-1} = 0$$

$$\Rightarrow 4c_{n+1}(n+1)n + 2(n+1)c_{n+1} = -c_n$$

$$\Rightarrow c_{n+1}\left[4n(n+1) + 2(n+1)\right] = -c_n$$

$$\Rightarrow \boxed{c_{n+1} = -\frac{c_n}{2(n+1)(2n+1)}}$$

$$\Rightarrow c_1 = -\frac{c_0}{2}\ ;\ c_2 = \frac{1}{4!}c_0\ ;\ c_3 = -\frac{c_2}{6\times5} = -\frac{c_0}{6\times5\times4!} \Rightarrow c_{n+1} = (-)^{n+1}\frac{c_0}{(2n+2)!}$$

$$\Rightarrow c_n = \frac{(-1)^n}{(2n)!}c_0$$

So the solution is: $\boxed{y_1(x) = \sum_{n=0}^\infty c_n x^{n+r_1} = c_0\sum_{n=0}^\infty \frac{(-1)^n}{(2n)!}x^n}$

and if we compare to the taylor series of the cosine function at 0 :

$$\cos x \simeq \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

then we can write: $\boxed{y_1(x) = C_0 \cos(\sqrt{x})}$

* case $r_2 = \frac{1}{2}$

the recurrence is: $C_{n+1} = \dfrac{-C_n}{(2n+2)(2n+3)}$

giving : $C_n = \dfrac{(-1)^n}{(2n+1)!} C_0$

and a second solution: $y_2(x) = C_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{n+\frac{1}{2}}$

$\left(x^{\frac{1}{2}}\right)^{2n+1}$

which expresses as: $\boxed{y_2(x) = C_0 \sin(\sqrt{x})}$

d'où la solution complète of the equation:

$$\boxed{y(x) = A \cos(\sqrt{x}) + B \sin(\sqrt{x})}$$

y_1 and y_2 are the 2 basis solution of the ED,
their wronskian is $-\frac{1}{2\sqrt{x}} \neq 0$.

NB: the function $y_1(x)$ is regular at $x=0$, but its first derivative $y'(x) = \frac{1}{2\sqrt{x}} \left(\cos\sqrt{x} - \sin\sqrt{x}\right)$ is singular at $x=0$

The method always give the first solution y_1 but y_2 is not found if

→ the roots r_1, r_2 are equal (double root)

→ $r_2 - r_1$ is integer, positive (i.e $r_2 > r_1$)

|| in this case we need another second solution ||

→ wronskian ?

→ reduction of order ?

→ or direct form, as follows.

* Form of the second solution if $r_1 = r_2$

let $y_1(x)$ the solution corresponding to r_1

→ same reasonning that we made for constant coef. equation:

change $r_2 \neq r_1$ and take the limit $r_2 \to r_1$

let $y_2(x)$ the sol. corresponding to y_2.

then $\dfrac{y_2 - y_1}{r_2 - r_1}$ is a solution (linear combination)

and tends towards $\tilde{y}_2 = \dfrac{\partial}{\partial r}(y_1)\Big|_{r=r_1}$

if y_1 is a Frobenius series:

$$y_1(x) = \sum_{n=0}^{\infty} C_n x^{n+r_1}$$

then $y_2(x) = \sum_{n=0}^{\infty} C'_n x^{n+r_2}$ if $r_1 \neq r_2$

the C_n^* are different

$\Rightarrow$ C_n is function of r

so $$\tilde{y}_2(x) = \frac{\partial}{\partial r}\left[\sum_{n=0}^{\infty} C_n x^{n+r}\right]$$

$$= \sum_{n=0}^{\infty} C'_n(r) \cdot x^{n+r} + C_n \frac{\partial}{\partial r} e^{(n+r)\ln x}$$

prendre $x = |x| e^{i\pi}$ si <0

$$= \sum_{n=0}^{\infty} b_n(r)\, x^{n+r} + \ln(x) \underbrace{\sum_{n=0}^{\infty} C_n x^{n+r}}_{y_1}$$

thus: $$\boxed{\tilde{y}_2(x) = \ln(|x|) \cdot y_1(x) + \underbrace{\sum_{n=0}^{\infty} b_n \cdot x^{n+r_1}}_{\text{Frobenius series}}}$$

cf. Spotlight....pdf

* In the case where $r_2 - r_1$ is an integer > 0.

search a solution of the form:

smallest

$$\tilde{y}_2 = \alpha\, y_1(x) \cdot \ln|x| + x^{r_1} \sum_{n=0}^{\infty} b_n x^n \quad \text{with } b_0 = 1$$

i.e. the same form than $r_1 = r_2$ but with a factor α $\ln|x|$ instead of $\ln|x|$.

why is it a problem : ex if $r_1 = -1$ and $r_2 = 2$.

$$y_1 = c_0 x^{-1} + c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 x^4$$

$$y_2 = c_0 x^2 + c_1 x^3 + c_2 x^4$$

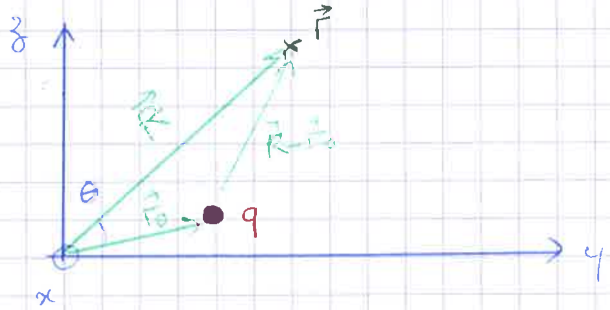
Chap 2 : Special Functions

I_ Legendre polynomials

1.) Historical introduction

Discovered (introduced) by Legendre (1752-1834) to express the potential gravitational. The problem is the same in electrostatics with the electric potential,

→ Consider a point-charge q at $\vec{r}_0$, and express the potential at $\vec{R}$, with $|\vec{R}|>|\vec{r}_0|$



$$V(\vec{r}) = \frac{q}{4\pi\varepsilon_0 |\vec{R}-\vec{r}_0|}$$

with : $|\vec{R}-\vec{r}_0| = \sqrt{(\vec{R}-\vec{r}_0)^2} = \sqrt{R^2 - 2\vec{R}\cdot\vec{r}_0 + r_0^2}$

$$= R\sqrt{1 - 2\frac{r_0}{R}\cos\theta + \left(\frac{r_0}{R}\right)^2}$$

let $r = \frac{r_0}{R}$ and $x = \cos\theta$ we have $|x| \leq 1$

⇒ the potential involves functions of type

$$\left[1 - 2rx + r^2\right]^{-\frac{1}{2}}$$

We want to make a series expansion at the origin and write

$$\boxed{\left[1 - 2rx + r^2\right]^{-\frac{1}{2}} = \sum_{n=0}^{\infty} r^n \times P_n(x)} \quad (1)$$

multipolar expansion.

$P_n(x)$ are the coefficients of the Taylor series.

the function $[1-2rx+r^2]^{-1/2}$ is called the generating function for Legendre polynomials.

2) Rodrigues Formula

the Taylor developpement of (1) gives

$$[1 - \underbrace{r(2x-r)}_{\varepsilon}]^{-\frac{1}{2}} \simeq 1 - \frac{\varepsilon}{2} + \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\varepsilon^2 + \cdots = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2}\varepsilon^n$$

↑ $r(2x-r)$

$\varepsilon < 1$ since $x<1$, $r<1$

$$= \sum_0^{\infty} \frac{(2n)!}{2^n (n!)^2} r^n (2x-r)^n$$

SF2

Expand the term $(2x-r)^n$

$$(2x-r)^n = \sum_{k=0}^{n} (-)^k C_n^k (2x)^{n-k} r^k$$

(formule du binôme) $(a+b)^n$

plug into the series, to obtain:

$$[1-r(2x-r)]^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, r^n$$

with
$$P_n(x) = \sum_{k=0}^{E(n/2)} \frac{(-)^k (2n-2k)!}{2^n k!\,(n-k)!\,(n-2k)!} x^{n-2k}$$

$|x| \leq 1$

Hence the functions $P_n(x)$ are polynomials of degree n

→ using the identity $\frac{d^n}{dx^n}(x^p) = \frac{p!}{(p-\ell)!} x^{p-\ell}$ we obtain:

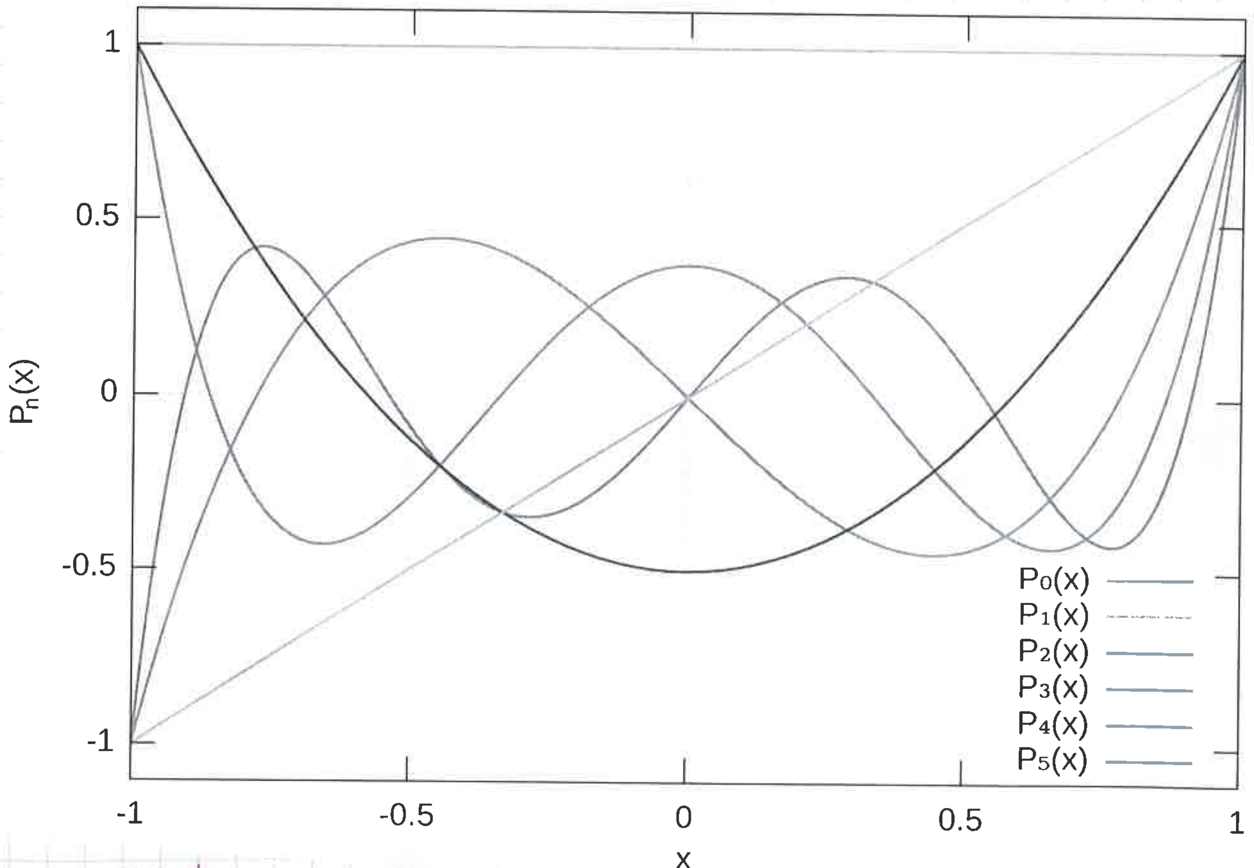
$$\boxed{P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\left[(x^2-1)^n\right]}$$

Rodrigues Formula

and the first polynomials are:

$P_0(x) = 1$ $\qquad$ $P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2-1)$ $\qquad$ $P_3(x) = \frac{1}{2}(5x^3-3x)$



montrer avec la tablette

* Some basic properties:

- $P_n(x)$ has the parity of n

  (consequence of Rodrigues formula)

- odd P_n are null at $x=0$

- $P_n(1) = 1 \quad \forall n$.

  it is a consequence of the expansion of the generating function at $x=1$:

$$\left[1-2r+r^2\right]^{-1/2} = (1-r)^{-1} = \sum_{n=0}^{\infty} \underset{\uparrow\, P_n(1)=1}{r^n} \qquad \text{(geom. series)}$$

3) Orthogonality.

The Legendre polynomials are orthogonal, with the scalar product defined as

$$\langle f, g\rangle = \int_{-1}^{1} f(x)\cdot g(x)\, dx.$$

this quantity has the properties of a scalar product between the functions f and g.

We shall see that $\langle P_m, P_n\rangle = 0$ if $m \neq n$

Proof: let consider $m<n$

$$\langle P_m, P_n\rangle = \frac{1}{2^n n!}\int_{-1}^{1} P_m(x)\cdot \frac{d^n}{dx^n}(x^2-1)^n\, dx. \qquad \text{integrate by parts.}$$

let $\varphi(x) = (x^2-1)^n$

$$\Rightarrow \langle P_m, P_n\rangle = \frac{1}{2^n n!}\int_{-1}^{1} P_m(x)\,\varphi^{(n)}(x)\, dx.$$

$$= \frac{1}{2^n n!}\left[P_m(x)\cdot\varphi^{(n-1)}(x)\right]_{-1}^{1} - \frac{1}{2^n n!}\int_{-1}^{1} P_m'(x)\cdot\varphi^{(n-1)}(x)\, dx$$

general result:

$$\int_{-1}^{1} f(x)\cdot\frac{d^n}{dx^n}(x^2-1)^n = (-1)^n\int_{-1}^{1} f^{(n)}(x)\,(x^2-1)^n\, dx$$

$\left[(x^2-1)^n\right]' \propto (x^2-1)^{n-1}$

$\left[(x^2-1)^n\right]'' \propto (x^2-1)^{n-2}$

$\vdots$

$\left[(x^2-1)^n\right]^{(n-1)} \propto (x^2-1) \Rightarrow$ null at $x=0,1$

therefore the [ ] is 0

So $\langle P_n, P_m \rangle = -\frac{1}{2^n n!} \int_{-1}^{1} \varphi^{(n-1)}(x)\, P_m'(x)\, dx$

$= +\frac{1}{2^n n!} \int_{-1}^{1} \varphi^{(n-2)}(x)\, P_m''(x)\, dx$ (1)

etc

until $P_m^{(m+1)}$ which is 0

$\Rightarrow \langle P_m, P_n \rangle = 0$ if $m < n$

Swap between m and n if not $\Rightarrow$ $\boxed{\langle P_m, P_n \rangle = 0 \text{ if } m \neq n.}$

$\rightarrow$ the norm of P_n : $\langle P_n, P_n \rangle$

(1) $\Rightarrow$ $\langle P_n, P_n \rangle = \frac{1}{2^n n!} \int_{-1}^{1} \varphi^{(n-n)}(x) \cdot P^{(n)}(x)\, dx$

$\varphi^{(0)}(x) = \varphi(x)$
$= (x^2-1)^n$

$\frac{1}{2^n n!} \varphi^{(2n)}(x)$
$\Downarrow$

$P^{(n)}$: $\varphi(x) = (x^2-1)^n = [x^{2n} + \dots]$

Tall this vanish at the derivative 2n

$\varphi' = 2n\, x^{2n-1} + \dots$

$\varphi'' = 2n(2n-1)\, x^{2n-2} + \dots$

$\Rightarrow \varphi^{(2n)} = (2n)!\, x^0$

$\Rightarrow \langle P_n, P_n \rangle = \frac{(2n)!}{(2^n n!)^2} \int_{-1}^{1} (x^2-1)^n\, dx.$

$\hookrightarrow$ use variable change $x = \cos\theta$

and use the relation

$$\frac{2}{2^{2n-1}} \int_0^{\pi/2} (\sin\theta)^{2n+1}\, d\theta = \frac{(n!)^2}{(2n+1)!}$$

(see properties of the Gamma function)

finally $\int_{-1}^{1} P_n(x)\, P_n(x)\, dx = \frac{2}{2n+1}$

d'où

$$\boxed{\langle P_n, P_m \rangle = \int_{-1}^{1} P_n(x)\, P_m(x)\, dx = \frac{2\,\delta_{mn}}{2n+1}}$$

(SF6)

* other families of polynomials: ¡à faire après le suivant!

- Gegenbauer, defined by a generating function

$$[1-2rx+r^2]^{-\alpha} = \sum_{n=0}^{\infty} C_n^{\alpha}(x)\, r^n$$

(generalisation of Legendre)

defined on $[-1, 1]$.

- Hermite polynomials: defined on $\mathbb{R}$. (application in quantum mechanics)

generating function:

$$\exp(2rx - r^2) = \sum_{n=0}^{\infty} H_n(x) \frac{r^n}{n!}$$

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$ equivalent to Rodrigues

- Laguerre $L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n}\left(e^{-x} x^n\right)$ defined on $\mathbb{R}+$

and some others: Chebyshev, Jacobi,

all these polynomial are orthogonal, with a scalar product

$$\langle f,g\rangle = \int_{\mathcal{D}} \rho(x)\, f(x).g(x)\, dx$$

$\mathcal{D}$ is the domain of definition of the polynom

$\rho(x)$ is a "density": 1 for Legendre

$e^{-x^2/2}$ for Hermite

$\frac{1}{\sqrt{1-x^2}}$ for Chebyshev

etc ...

* Expansion of a function on the Legendre polynomial

the family $\{P_n\}$ forms a complete basis set of the space of function which verify $\int_{-1}^{1} f(x)^2\, dx$ exists

therefore, any function meeting this criterium can be written as a series:

$$\boxed{f(x) = \sum_{n=0}^{\infty} c_n P_n(x)}$$

Fourier - Legendre series

equivalent to Fourier series: $f(t) = \sum c_n e^{2i\pi n \frac{t}{T}}$ if f is T-periodic the c_n are given by :

$$\boxed{c_n = \frac{2n+1}{2} \langle f(x), P_n \rangle = \frac{2n+1}{2} \int_{-1}^{1} f(x).\, P_n(x) . dx}$$

4) Some recurrence relations

all deduced from Rodrigues formula.

- various degrees :

$$n\, P_n(x) = (2n-1)\, x\, P_{n-1}(x) - (n-1)\, P_{n-2}(x)$$

- derivatives :

$$(x^2-1)\, P_n'(x) = n\left(x\, P_n(x) - P_{n-1}(x) \right)$$

$$x\, P_n'(x) - P_{n-1}'(x) = n\, P_n(x)$$

$$(2n+1)\, P_n = P_{n+1}' - P_{n-1}'$$ interesting for $\int P_n . dx$.

5) Legendre differential equation.

also deduced from Rodrigues formula, through the recurrence relations.

$$\boxed{(1-x^2)\, P_n''(x) - 2x\, P_n'(x) + n(n+1)\, P_n(x) = 0}$$

or equivalently :

$$\boxed{\frac{d}{dx}\left[(1-x^2)\, P_n' \right] + n(n+1)\, P_n = 0}$$

(vidéos sur youtube)

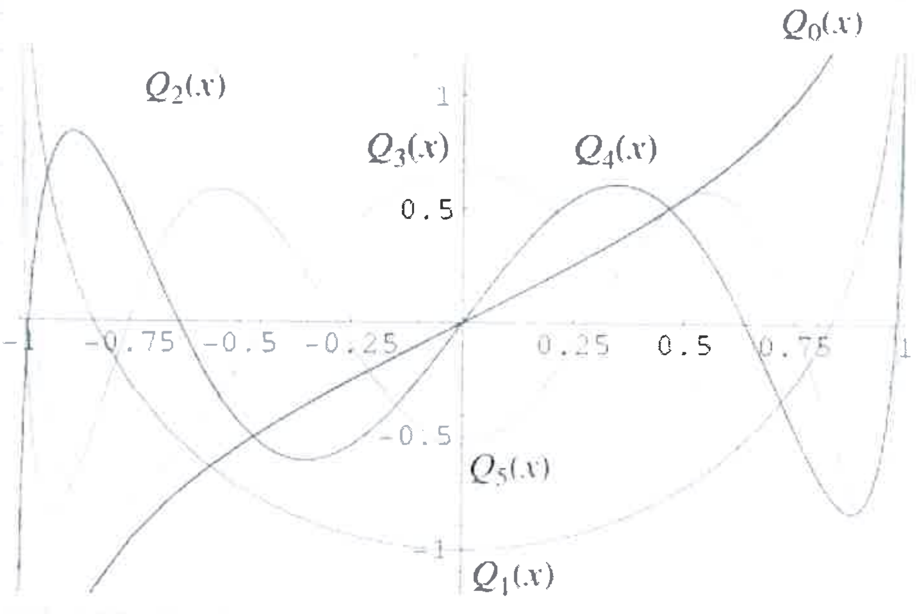
Frobenius method can be used to solve the equation.

It gives indeed two independent solutions:

→ $P_n(x)$, regular on $[-1, 1]$

→ $Q_n(x)$, singular at ± 1

= Legendre functions of the 2^{nd} kind.



$$Q_0 = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$$

$$Q_1 = x\, Q_0 - 1$$

$$Q_2 = P_2 \cdot Q_0 - \frac{3x}{2}$$

6) Associated Legendre functions P_ℓ^m

definition:

$$\boxed{P_\ell^m(x) = (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_\ell(x)} \qquad -\ell \leq m \leq \ell$$

with $P_\ell^0(x) = P_\ell(x) \Rightarrow P_\ell^m$ is a generalisation of P_ℓ.

The properties of the P_ℓ can be generalised as follows:

- Rodrigues formula: $$P_\ell^m(x) = (1-x^2)^{\frac{m}{2}} \frac{1}{2\ell!} \frac{d^{\ell+m}}{dx^{\ell+m}} (x^2-1)^\ell$$

- Parity : $\ell + m$

- relation between P_ℓ^m and P_ℓ^{-m} :

$$\boxed{P_\ell^{-m}(x) = (-)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(x)}$$

the demo comes from the expression of P_ℓ page SF2 (the sum with x^{n-2k}). See Ayant, "Fonctions spéciales"

- orthogonality: (for fixed m)

$$\int_{-1}^{1} P_\ell^m P_n^{*m}\, dx = \frac{2(\ell+m)!}{(2\ell+1)(\ell-m)!} \delta_{\ell n}$$

fixed ℓ:

$$\int_{-1}^{1} P_\ell^m P_\ell^n = \frac{(\ell+m)!}{m(\ell-m)!} \delta_{mn}$$

- Differential equation :

also exists

$$\boxed{(1-x^2)\, y'' - 2x\, y' + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right] y = 0}$$

solution regular at $x=1$ if $|m| \leq \ell$

7) Solutions of Laplace equation in spherical coordinates.

Laplace equation for a function $V(r, \theta, \varphi)$ is

$$\Delta V = 0$$

$$\Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial\varphi^2} = 0$$

we search for a separable solution of the form

$$V(r,\theta,\varphi) = f(r) \times g(\theta,\varphi) \qquad \text{with } g = F(\theta).G(\varphi)$$

$$\Rightarrow \frac{g}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{f}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial g}{\partial\theta}\right) + \frac{f}{r^2\sin^2\theta}\frac{\partial^2 g}{\partial\varphi^2} = 0$$

divide by $\frac{fg}{r^2}$

$$\underbrace{\frac{1}{f}\frac{d}{dr}\left(r^2\frac{df}{dr}\right)}_{\text{function of } r \text{ alone}} = \underbrace{-\frac{1}{g\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial g}{\partial\theta}\right] - \frac{1}{g.\sin^2\theta}\frac{\partial^2 g}{\partial^2\varphi}}_{\text{function of } \theta,\varphi \text{ alone}}$$

possible only if the lhs and rhs are constants
(i.e. indep. of r / θ,φ)

we put this for convenience

$$\Rightarrow \begin{cases} \dfrac{1}{f}\dfrac{d}{dr}\left(r^2\dfrac{df}{dr}\right) = cte = \alpha(\alpha+1). & \alpha \in \mathbb{R} \\[2ex] \dfrac{1}{g\sin\theta}\dfrac{\partial}{\partial\theta}\left[\sin\theta\dfrac{\partial g}{\partial\theta}\right] + \dfrac{1}{g\sin^2\theta}\dfrac{\partial^2 g}{\partial^2\varphi} = -\alpha(\alpha+1) \end{cases}$$

* Solutions for the radial eq.

$$\frac{d}{dr}\left(r^2\frac{df}{dr}\right) = \alpha(\alpha+1)\, f$$

$$\Rightarrow r^2 f'' + 2r f' - \alpha(\alpha+1) f = 0$$

euler eq.: sol^n of the forme x^λ

SF 10

we find 2 values of d: $\lambda_1 = \alpha$, $\lambda_2 = -\alpha - 1$

So: $$\boxed{f(r) = A_1 r^{\alpha} + \frac{A_2}{r^{\alpha+1}}}$$

* Solutions for the angular eq.

let $g(\theta, \varphi) = F(\theta) . G(\varphi)$

$$\Rightarrow \frac{\cancel{G}}{F.\cancel{G} \sin\theta} \frac{\partial}{\partial\theta}\left[\sin\theta \frac{\partial F}{\partial\theta}\right] + \frac{\cancel{F}}{\cancel{F}.G.\sin^2\theta} \frac{d^2 G}{d\varphi^2} = -\alpha(\alpha-1)$$

$\times \sin^2\theta$

$$\underbrace{\frac{\sin\theta}{F}\frac{d}{d\theta}(\sin\theta . F') + \alpha(\alpha+1)\sin^2\theta}_{\text{fn of } \theta \text{ alone}} = \underbrace{-\frac{G''}{G}}_{\text{fn of } \varphi \text{ alone}}$$

each is a constant $= \lambda \in \mathbb{R}$

$$\Rightarrow \begin{cases} \dfrac{G''}{G} = -\lambda \\ \dfrac{\sin\theta}{F}\dfrac{d}{d\theta}(F'\sin\theta) + \alpha(\alpha+1)\sin^2\theta = \lambda \end{cases}$$

→ eq. in φ:

$G'' + \lambda G = 0$ harmonic

periodic in φ if $\lambda > 0$

$\Rightarrow$ let $\lambda = m^2$

$$\Rightarrow \boxed{G(\varphi) = B_1 e^{im\varphi} + B_2 e^{-im\varphi}}$$

$\frac{2\pi}{m}$ periodic

m entier for 2π periodicity

→ eq in θ

$$\sin\theta . \frac{d}{d\theta}(F'\sin\theta) + [\alpha(\alpha+1)\sin^2\theta - m^2] F = 0$$

$/\sin^2\theta$

$$\Rightarrow \frac{1}{\sin\theta}\frac{d}{d\theta}(F'\sin\theta) + \left[\alpha(\alpha+1) - \frac{m^2}{\sin^2\theta}\right] F = 0$$

$F''\sin\theta + F'\cos\theta$

$$\Rightarrow F'' + F'\frac{\cos\theta}{\sin\theta} + \left[\alpha(\alpha+1) - \frac{m^2}{\sin^2\theta}\right] F = 0$$

SF 11

let $u = \cos\theta$ $\Rightarrow$ $\frac{dF}{d\theta} = \frac{dF}{du}\cdot\frac{du}{d\theta} = -\frac{dF}{du}\sin\theta$

$\sin\theta = \pm\sqrt{1-u^2}$

$$\frac{d^2F}{d\theta^2} = -\sin\theta \cdot \frac{d}{du}\left(\frac{dF}{du}\right)\cdot\frac{du}{d\theta} - \cos\theta\,\frac{dF}{du}$$

$$= \sin^2\theta\,\frac{d^2F}{du^2} - \cos\theta\,\frac{dF}{du}$$

plug into the D.E to obtain:

$$(1-u^2)\frac{d^2F}{du^2} - 2u\frac{dF}{du} + \left[\alpha(\alpha+1) - \frac{m^2}{1-u^2}\right]F = 0$$

it can be shown that the solutions for α not integer diverge at $u = -1$ ($\theta = \pi$)

$\Rightarrow$ physically acceptable solutions correspond to $\alpha = \ell \in \mathbb{N}$ with $|m| \leq \ell$

Solutions to the above eq. for $\alpha = \ell$ integer are $P_\ell^m(u)$ and $P_\ell^{-m}(u)$ since m is squared

$\Rightarrow$ that gives the form:

$$F = c_1 P_\ell^m(u) + c_2 P_\ell^{-m}(u)$$

$$\Rightarrow F(\theta) = c_1 P_\ell^m(\cos\theta) + c_2 P_\ell^{-m}(\cos\theta)$$

$\Rightarrow$ the angular function $g(\theta,\varphi) = F(\theta)\,G(\varphi)$ is:

$$g(\theta,\varphi) = \left(B_1 e^{im\varphi} + B_2 e^{-im\varphi}\right)\left(c_1 P_\ell^m(\cos\theta) + c_2 P_\ell^{-m}(\cos\theta)\right)$$

since $P_\ell^{-m} \propto P_\ell^m$, we can write:

$$g(\theta,\varphi) = \alpha\, P_\ell^m(\cos\theta)\, e^{im\varphi} + \beta\, P_\ell^{-m}(\cos\theta)\, e^{-im\varphi}$$

$g \in \mathbb{R}$

We introduce the function

spherical harmonic

$$\boxed{Y_\ell^m(\theta,\varphi) = \underbrace{\left[\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}\right]^{1/2}}_{\text{normalisation constant}} P_\ell^m(\cos\theta)\, e^{im\varphi}}$$

so that $\langle Y_\ell^m, Y_\ell^m\rangle = 1$

taking $f(r)$ and $g(\theta,\varphi)$ together, the solutions for the Laplace equation are of the form:

$$v_{\ell,m}(r,\theta,\varphi) = \left(A_1 r^{\ell} + \frac{A_2}{r^{\ell+1}}\right)\left(D_1 Y_\ell^m(\theta,\varphi) + D_2 Y_\ell^{-m}(\theta,\varphi)\right)$$

there is one solution for every couple (ℓ, m). So the linear combination of the individual solutions $v_{\ell m}(\theta,\varphi)$ gives the final form:

$$\boxed{V(r,\theta,\varphi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell}\left[A_{\ell m} r^{\ell} + \frac{B_{\ell m}}{r^{\ell+1}}\right] Y_\ell^m(\theta,\varphi)}$$

no term Y_ℓ^{-m} since the sum starts at $-\ell$

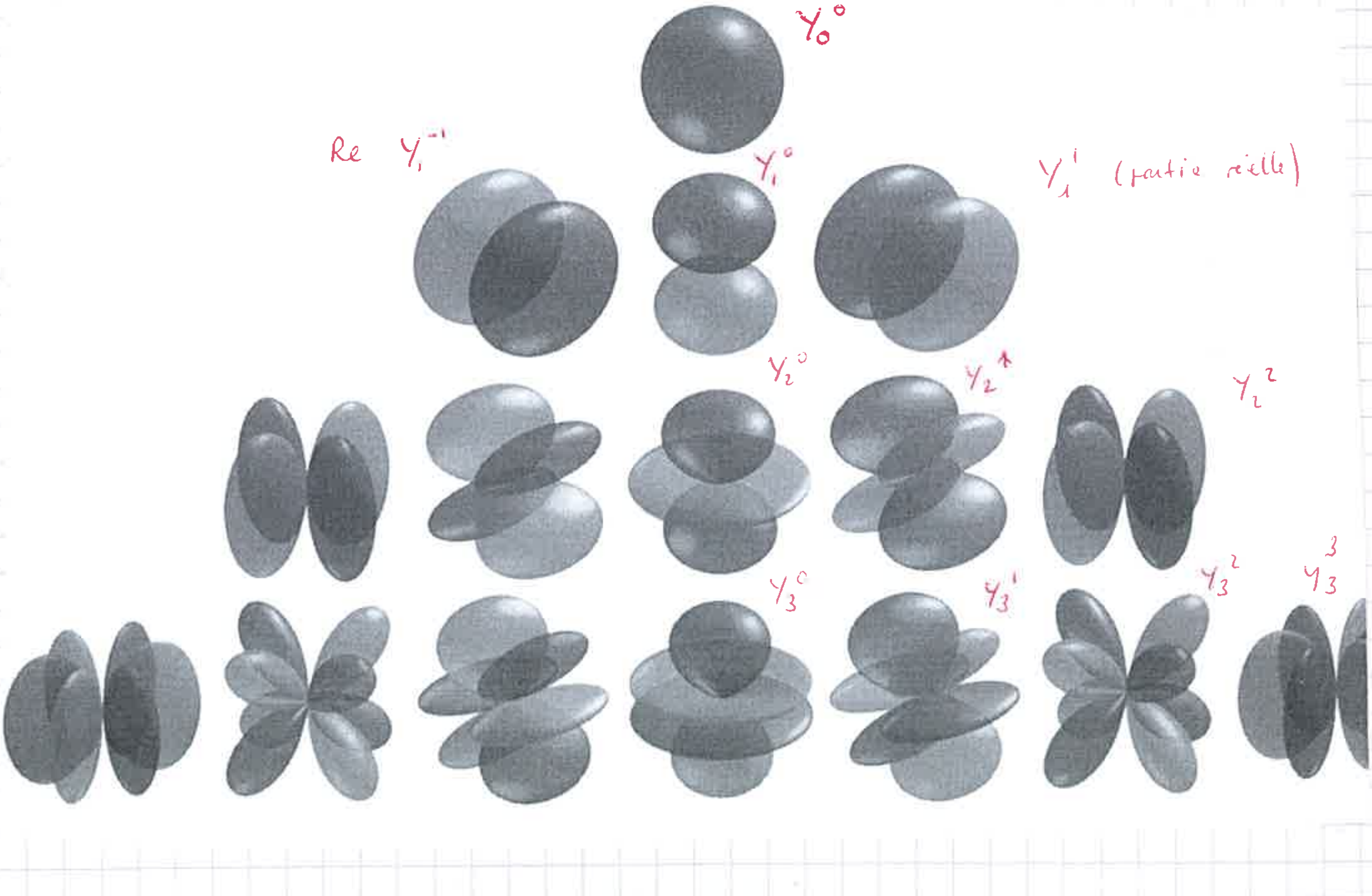
8) The Spherical harmonics Y_ℓ^m

a) The first ones

$$Y_0^0(\theta,\varphi) = \frac{1}{\sqrt{2\pi}}$$

$$Y_1^{-1}(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta\, e^{-i\varphi} = -\overline{Y_1^1(\theta,\varphi)} \quad ; \quad Y_1^0(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta$$

$$Y_2^0(\theta,\varphi) = \frac{1}{4}\sqrt{\frac{5}{\pi}}(3\cos^2\theta - 1) \quad ; \quad Y_2^1(\theta,\varphi) = -\frac{1}{2}\sqrt{\frac{15}{2\pi}}\sin\theta.\cos\theta.e^{i\varphi} \quad ; \quad Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\varphi}$$



★ basic properties :

$$\boxed{Y_\ell^{-m}(\theta,\varphi) = (-1)^m\, \overline{Y_\ell^m(\theta,\varphi)}} \qquad \text{parity}$$

$$\boxed{Y_\ell^m(\pi-\theta, \varphi+\pi) = (-1)^\ell\, Y_\ell^m(\theta,\varphi)} \qquad \text{symmetry.}$$

b) Orthogonality

As for the Legendre polynomials, we define a scalar product for functions $f(\theta,\varphi)$ on the unit sphere :

$$\langle f, g\rangle = \int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} f(\theta,\varphi)\cdot \overline{g(\theta,\varphi)}\cdot \sin\theta\, d\theta\, d\varphi$$

It can be proved that the Y_ℓ^m are orthonormal, i.e. :

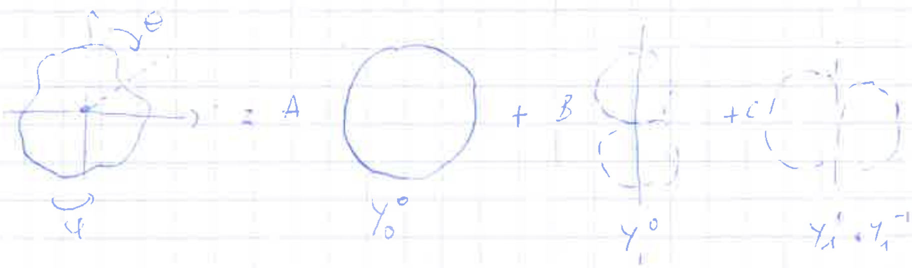
$$\boxed{\langle Y_\ell^m, Y_{\ell'}^{m'}\rangle = \iint_{\text{Sphere}} Y_\ell^m(\theta,\varphi)\cdot \overline{Y_{\ell'}^{m'}(\theta,\varphi)}\ \sin\theta\, d\theta\, d\varphi = \delta_{\ell\ell'}\,\delta_{mm'}}$$

It can also be proved that the Y_ℓ^m form a complete basis of the functions $f(\theta,\varphi)$, so that any function square summable over the sphere can be written as :

$$f(\theta,\varphi) = \sum_{n=0}^{\infty}\sum_{m=-\ell}^{\ell} a_{\ell m}\ Y_\ell^m(\theta,\varphi)$$

with $\quad a_{\ell m} = \langle f, Y_\ell^m\rangle = \iint_{\text{Sphere}} f(\theta,\varphi)\ \overline{Y_\ell^m(\theta,\varphi)}\ \sin\theta\, d\theta\, d\varphi$

for example the terrestrial geoid of radius $R(\theta,\varphi)$:



We have also :

$$\frac{1}{|\vec r - \vec r\,'|} = \sum_{\ell, m} \frac{4\pi}{2\ell+1}\,\frac{r_<^\ell}{r_>^{\ell+1}}\ Y_\ell^m(\theta,\varphi)\ \overline{Y_{\ell m}(\theta',\varphi')} \qquad r_< = \min\{|\vec r|, |\vec r\,'|\}$$

$\vec r : (r,\theta,\varphi)$

(equivalent to the multipole expansion)

$\vec r\,' : (r',\theta',\varphi')$

SF14

II _ Bessel Functions

1) Bessel differential equation

$$\boxed{x^2 y'' + x y' + (x^2 - \nu^2) y = 0}$$

2 basis functions : $\begin{cases} J_\nu(x) & \text{first kind} \\ Y_\nu(x) = N_\nu(x) & \text{second kind.} \end{cases}$
(Neumann ou Weber).

We will solve the D.E. by Frobenius method

search for a series : $y = x^r \sum_{n=0}^{\infty} c_n x^n$

- $y = \sum_0 c_n x^{n+r}$
- $x y' = \sum_0 c_n (n+r) x^{n+r}$
- $x^2 y'' = \sum_0 (n+r)(n+r-1) c_n x^{n+r}$
- $x^2 y = \sum_0 c_n x^{n+r+2} = \sum_2 c_{n-2} x^{n+r}$

$$= \sum_0 c_n x^{n+r} \left[(n+r)(n+r-1) - (n+r) - \nu^2 \right] + \sum_2 c_{n-2} x^{n+r} = 0$$

term $n=0$: $\quad c_0 x^r \left[(r-1) r + r - \nu^2 \right] = 0 \quad$ eq. indicial.

si $c_0 \neq 0 \quad \Rightarrow \quad r^2 = \nu^2 \quad \Rightarrow \quad \boxed{r = \pm \nu}$

we denote ν the positive root :

$\nu \geq 0$

term $n=1$: $\quad c_1 \left[(1+r)^2 - \nu^2 \right] = 0 \quad$ since $\nu^2 = r^2$

$\Rightarrow \quad \boxed{c_1 = 0}$

* Case $r = +\nu$

recurrence : $\quad c_n = - \dfrac{c_{n-2}}{n(n+2\nu)} \quad$ with n even $\quad (c_1 = 0)$

we obtain : $\quad \boxed{c_{2k} = \dfrac{(-)^k a_0 \, \nu!}{2^{2k} k! \, (\nu+k)!}}$

a_0 is an arbitrary constant $\quad$ (for the definition of J_ν we take $a_0 = \dfrac{1}{2^\nu \nu!}$)

and the first basis solution is defined as :

$$\boxed{J_\nu(x) = \sum_{k=0}^{\infty} (-)^k \frac{1}{k!\,(\nu+k)!} \left(\frac{x}{2}\right)^{\nu+2k}} \quad \nu \in \mathbb{R}_+$$

regular at $x=0$

with the real factorial $x! = \Gamma(x+1) = \int_0^\infty t^x e^{-t}\,dt$

* case $r = -\nu$

the recurrence is: $c_n = - \dfrac{c_{n-2}}{n(n-2\nu)}$

and the series is :

$$J_{-\nu}(x) = \sum_{k=0}^{\infty} (-)^k \frac{1}{k!\,(k-\nu)!} \left(\frac{x}{2}\right)^{2k-\nu} \quad \begin{pmatrix} \nu > 0 \\ -\nu < 0 \end{pmatrix}$$

singular at $x=0$

which is the same formula, therefore valid $\forall \nu \notin \mathbb{Z}^*_-$ (in that case the solution diverges)

[if ν is not integer the solutions J_ν and $J_{-\nu}$ are linearly independent and the general solution to Bessel's D. E. is the linear combination

$$Z_\nu(x) = a\,J_\nu(x) + b\,J_{-\nu}(x)$$]

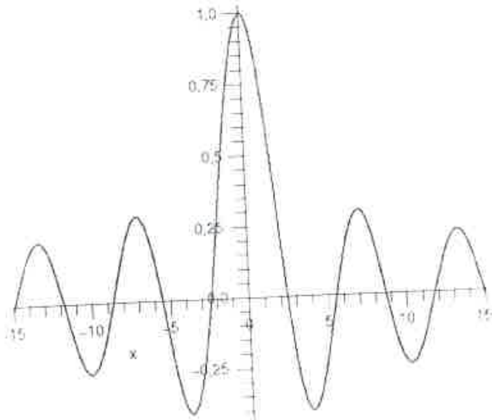
2) Bessel Function of the first kind

$J_\nu(x)$ defined by the above relation.

a) Function $J_0(x)$

$$J_0(x) = \sum_{k=0}^{\infty} (-)^k \frac{1}{(k!)^2} \frac{x^{2k}}{2^{2k}}$$

even function



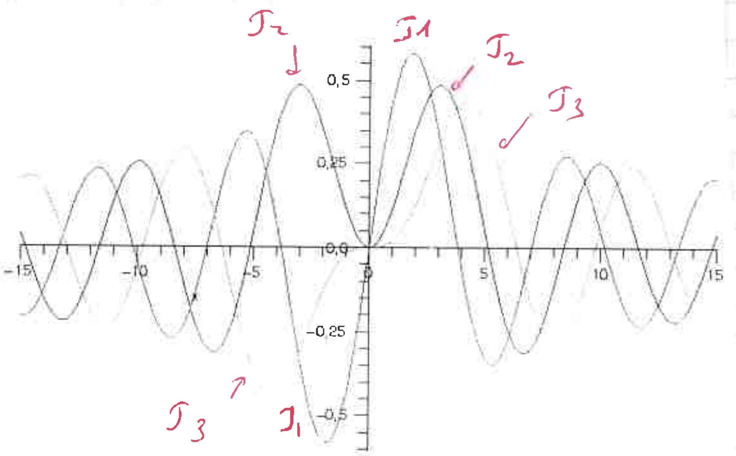
at $x \simeq 0$:

$J_0(x) \simeq 1 - \dfrac{x^2}{4} + o(x^4)$

1st zero at $x_0 = 2,4048\ldots$

SF 1

b) Functions $J_n(x)$, n integer > 0



→ negative degree:

$$J_{-n}(x) = (-)^n J_n(x)$$

↳ explain why J_n and J_{-n} are not lin. indep.

→ parity: J_n has the parity of n

→ limit at $x \to 0$:

$$J_n(x) \to \frac{1}{n!}\left(\frac{x}{2}\right)^n$$

$$J_1 \sim \frac{x}{2}$$

$$J_2 \sim \frac{x^2}{8}$$

at $x \to \infty$ -

$J_n(x) \to \sqrt{\frac{\pi}{2x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)$

c) Functions $J_{n+\frac{1}{2}}(x)$, half integer index

In that case J_ν involves trigonometric functions.

We have: $$J_{\frac{1}{2}}(x) = \sum_{k\geq 0}^{\infty} (-)^k \frac{1}{k!\,(k+\frac{1}{2})!} \left(\frac{x}{2}\right)^{2k+\frac{1}{2}}$$

identity: $(n+\frac{1}{2})! = \frac{\sqrt{\pi}\,(2n+1)!}{2^{2n+1}\, n!}$

$$\Rightarrow J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \underbrace{\sum_{k\geq 0}^{\infty} (-)^k \frac{x^{2k+1}}{(2k+1)!}}_{\text{taylor series of } \sin}$$

$$\Rightarrow \boxed{J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x}$$

and

$$\boxed{J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x}$$

→ two basis solutions of Bessel's ED. for $\nu = \frac{1}{2}$

$J_{\frac{1}{2}}$ regular at 0, $J_{-\frac{1}{2}}$ has a singularity

It can be shown that

$$J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)$$ regular at 0

$$J_{-\frac{1}{2}}(x) = -\sqrt{\frac{2}{\pi x}}\left(\frac{\cos x}{x} + \sin x\right)$$ singular at 0

d) Recurrence relations.

We consider :

$$\frac{d}{dx}\left[\left(\frac{x}{2}\right)^{\nu} J_\nu(x)\right] = \sum_{k=0}^{\infty} (-)^k \frac{1}{k!\,(\nu+k)!}\,\frac{d}{dx}\left(\frac{x}{2}\right)^{2\nu+2k}$$

$$= \sum (-)^k \frac{1}{k!\,(\nu+k)!}\,\frac{1}{2^{2\nu+2k}}\,(2\nu+2k)\,x^{2\nu+2k-1}$$

$$= \sum \frac{(-)^k}{k!\,(\nu+k)!}\,(\nu+k)\left(\frac{x}{2}\right)^{2\nu+2k-1}$$

$$= \underbrace{\sum \frac{(-)^k}{k!\,(\nu+k-1)!}\left(\frac{x}{2}\right)^{\nu-1+2k}}_{J_{\nu-1}}\left(\frac{x}{2}\right)^{\nu} = \left(\frac{x}{2}\right)^{\nu} J_{\nu-1}(x)$$

$\Rightarrow$

$$\boxed{\frac{d}{dx}\left[x^{\nu} J_\nu(x)\right] = x^{\nu} J_{\nu-1}(x)}$$

and

$$\boxed{\frac{d}{dx}\left[x^{-\nu} J_\nu(x)\right] = -x^{-\nu} J_{\nu+1}(x)}$$

Hence :

$$\frac{d}{dx}\left(x J_1(x)\right) = x J_0(x)$$

useful in optics. (Airy pattern)

Other useful relations :

$$J_\nu(x) = \frac{x}{2\nu}\left[J_{\nu-1}(x) + J_{\nu+1}(x)\right]$$

$$J_\nu'(x) = \frac{1}{2}\left[J_{\nu-1}(x) - J_{\nu+1}(x)\right] \qquad \nu \neq 0$$

$$J_0'(x) = -J_1(x)$$

$$x J_\nu'(x) = \nu J_\nu(x) - x J_{\nu+1}(x)$$

e) Generating function of J_n

as for Legendre polynomials where

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, t^n$$

where $(\quad)^{-1/2}$ is the generating function of P_n, we have also a function $g(x,t)$ for the Bessel functions J_n:

$$\boxed{g(x,t) = \exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)\right] = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n}$$

proof: expand $g(x,t)$ as a Taylor series in t^n:

$$g(x,t) = \exp\frac{xt}{2} \cdot \exp -\frac{x}{2t} = \sum_{k=0}^{\infty} \frac{(xt)^k}{2^k k!} \cdot \sum_{\ell=0}^{\infty} \frac{(-x)^\ell}{2^\ell \ell!}$$

$$= \sum_{k,\ell} (-)^\ell \frac{1}{k!\,\ell!} \frac{x^{k+\ell}}{2^{k+\ell}} t^{k-\ell} \leftarrow n$$

$n = k-\ell \iff \ell = k-n \iff k+\ell = 2k-n$

n can be <0 or ≥ 0

- $n \geq 0$

$$g = \sum_{n=0}^{\infty} \underbrace{\left\{ \sum_{k=n}^{\infty} \frac{(-)^{k-n}}{k!\,(k-n)!} \left(\frac{x}{2}\right)^{2k-n} \right\}}_{\text{coef. } C_n \text{ of the Taylor series}} t^n$$

$$C_n = \sum_{k=n}^{\infty} \frac{(-)^{k-n}}{k!\,(k-n)!} \left(\frac{x}{2}\right)^{2k-n}$$

$\ell = k-n$

$$C_n = \sum_{\ell=0}^{\infty} \frac{(-)^\ell}{\ell!\,(\ell+n)!} \left(\frac{x}{2}\right)^{2\ell+n}$$

$\leftarrow$ expression of the series for $J_n(x)$

$$= J_n(x)$$

- for $n<0$: the change is the sum $\sum_k$ for C_n which starts at $k=0$ instead of $k=n$

we find $C_n = (-)^n J_{-n}(x) = J_n(x)$ as well.

therefore

$$g(x,t) = e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{+\infty} J_n(x)\, t^n$$

f) Integral form of $J_n(x)$

change $t = e^{i\varphi}$ into the generating function, we have

$$\exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)\right] = \exp\left[\frac{x}{2}\left(e^{i\varphi}-e^{-i\varphi}\right)\right]$$

$$\exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)\right] = \sum_{n=-\infty}^{+\infty} J_n(x)\, t^n = \sum_{n=-\infty}^{+\infty} J_n(x)\, e^{in\varphi}$$

$$\exp\left[\frac{x}{2}\left(e^{i\varphi}-e^{-i\varphi}\right)\right] = \exp\left[x\, i \sin\varphi\right]$$

Therefore we obtain a Fourier series:

$$\boxed{\underbrace{e^{ix\sin\varphi}}_{2\pi \text{ periodic in } \varphi} = \sum_{n=-\infty}^{+\infty} J_n(x)\, \underbrace{e^{in\varphi}}_{\text{harmonics of the frequency } \frac{1}{2\pi}\ \left(e^{2i\pi n \frac{t}{2\pi}}\right)}}$$

and since the Fourier coefficients a_n of the Fourier series of a function f with period T are given by

$$a_n = \frac{1}{T}\int_{-\frac{T}{2}}^{T/2} f(t)\cdot e^{-2i\pi n \frac{t}{T}}\, dt$$

we have:

$$\boxed{J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix\sin\varphi}\, e^{-in\varphi}\, d\varphi}$$

or equivalently

since $\sin\varphi = \cos\left(\varphi - \frac{\pi}{2}\right)$

$$J_n(x) = \frac{i^{-n}}{2\pi}\int_{-\pi}^{\pi} e^{ix\cos\varphi}\, e^{+in\varphi}\, d\varphi$$

equivalently, we can show that

$$\boxed{e^{ix\cos\varphi} = \sum_{n=-\infty}^{+\infty} i^n J_n(x)\, e^{in\varphi}}$$

g) The Hankel transform.

An application of Bessel functions is the 2D Fourier transform of radial functions $f(\rho)$:

$\rho = (x, y)$ in the plane

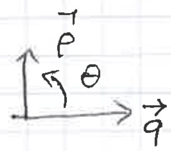
$$\hat f(u,v) = \iint_{-\infty}^{+\infty} f(x,y)\, e^{-2i\pi(ux+vy)}\, dx\, dy.$$

if f depends on $\rho = \sqrt{x^2+y^2}$ we have, in polar coords

$$\hat f(u,v) = \int_{\rho=0}^{\infty}\int_{\theta=0}^{2\pi} f(\rho)\, e^{-2i\pi \vec q\cdot\vec\rho}\, \rho\, d\rho\, d\theta$$

where $\vec q = \begin{vmatrix} u \\ v \end{vmatrix}$

θ can be chosen so that its origin is the q axis:



$$\Rightarrow \quad \hat f(u,v) = \int_{\rho=0}^{\infty}\int_{\theta=0}^{2\pi} f(\rho).\ e^{-2i\pi q\rho\cos\theta}\, \rho\, d\rho\, d\theta$$

$$= \int_{\rho=0}^{\infty} \rho f(\rho)\, d\rho \int_{\theta=0}^{2\pi} e^{-2i\pi q\rho\cos\theta}\, d\theta$$

$$J_0(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix\cos\theta}\, d\theta$$

$$\Rightarrow \quad \boxed{\hat f(u,v) = \tilde{f}(q) = 2\pi\int_0^{\infty} \rho f(\rho)\, J_0(2\pi q\rho)\, d\rho}$$

inverse transform:

$$\boxed{f(\rho) = 2\pi\int_0^{\infty} q\tilde f(q)\, J_0(2\pi q\rho)\, dq}$$

Applications

h) Orthogonality

→ see page SF 28

3) Bessel Function of second kind

Def:

$$\boxed{Y_\nu(x) = \frac{J_\nu(x)\cos(\pi\nu) - J_{-\nu}(x)}{\sin(\pi\nu)}} \qquad \nu \in \mathbb{R}$$

→ sometimes called Weber or Neumann function, $N_\nu(x)$

→ linear combination of J_ν and $J_{-\nu}$: solution of Bessel's D. Eq.

→ when ν is integer, say $\nu = n \in \mathbb{N}$:

$$\cos(\pi n) = (-)^n \quad ; \quad J_{-n} = (-)^n J_n$$

thus $Y_n(x) \rightarrow \frac{0}{0}$ undetermined

the undetermination is removed with L'Hôpital's rule:

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)} \qquad \text{if} \quad f(x_0) = g(x_0) = \begin{cases} 0 \\ \pm\infty \end{cases}$$

Hence $Y_n(x)$ is the limit:

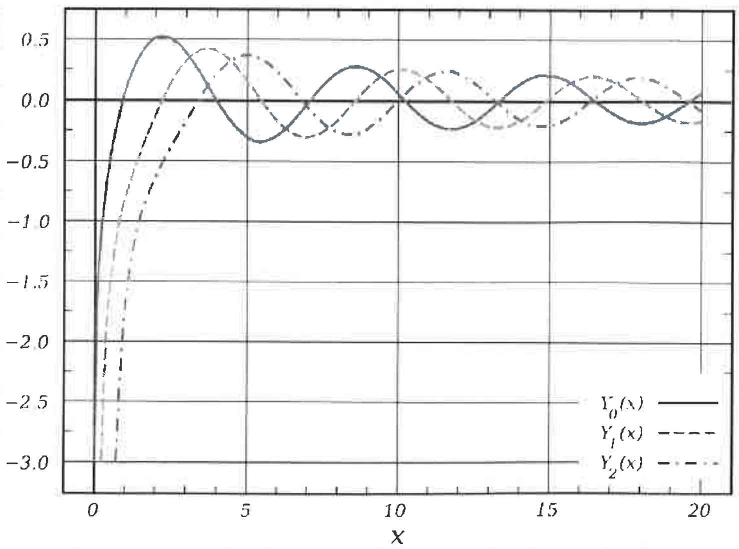
$$Y_n(x) = \frac{1}{\pi}\left[\frac{\partial J_\nu(x)}{\partial \nu} - (-)^n \frac{\partial J_{-\nu}(x)}{\partial \nu}\right]_{\nu = n}$$

It can be shown that Y_n is a solution of the D. Eq. for n integer.

⇒ $\{J_n, Y_n\}$ form the 2 basis solutions of Bessel's D. E. for n integer.

$Y_\nu(x)$ is singular at $x = 0$

and complex-valued at $x < 0$



* Some properties.

- The recurrence relations :

same as J_ν

$$\begin{cases} \frac{d}{dx}\left[x^{\nu} Y_\nu(x)\right] = x^{\nu} Y_{\nu-1}(x) \\ \frac{d}{dx}\left[x^{-\nu} Y_\nu(x)\right] = -x^{-\nu} Y_{\nu-1}(x) \end{cases}$$

$$\begin{cases} \frac{2\nu}{x} Y_\nu = Y_{\nu-1} + Y_{\nu+1} \\ 2 Y_\nu' = Y_{\nu-1} - Y_{\nu+1} \quad ; \quad Y_0' = -Y_1 \end{cases}$$

- half integer index :

$$J_{-(m+\frac{1}{2})} = (-)^{m+1} Y_{m+\frac{1}{2}}$$

$$Y_{-(m+\frac{1}{2})} = (-)^{m} J_{m+\frac{1}{2}}$$

- Small argument $x \simeq 0$:

asymptotic $x \to \infty$:
$$Y_\nu \sim \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

$$Y_\nu(x) \simeq -\frac{(\nu-1)!}{\pi} \left(\frac{2}{x}\right)^{\nu} \qquad \nu \neq 0$$

$$Y_0(x) \simeq \frac{2}{\pi}\left[\ln\left(\frac{x}{2}\right)\right]$$

γ = Euler constant $\simeq 0,577$

always diverges.

- integral form: (n integer)

$x > 0$

$$Y_n(x) = \frac{1}{\pi}\int_0^{\pi} \sin(x \sin\theta - n\theta)\, d\theta - \frac{1}{\pi}\int_0^{\infty}\left[e^{nt} + (-)^n e^{-nt}\right] e^{-x\, \mathrm{sh}\, t}\, dt$$

* Hankel functions

$$\begin{cases} H_\nu^{(1)}(x) = J_\nu(x) + i\, Y_\nu(x) \\ H_\nu^{(2)}(x) = J_\nu(x) - i\, Y_\nu(x) \end{cases}$$

4) Other Bessel functions

* I_ν : modified Bessel function of the first kind.

$$I_\nu(x) = i^{-\nu} J_\nu(ix) \qquad \nu \text{ real}$$

$\Rightarrow$ I_ν is a particular case of imaginary argument of J_ν

* K_ν : modified Bessel function of the second kind.

defined as :

$$K_\nu = \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin \pi\nu}$$

$$K_\nu(x) = \frac{\pi}{2} i^{\nu+1} \left[J_\nu(ix) + i\, Y_\nu(ix) \right]$$

both I_ν and K_ν are independent solutions of Bessel's D.E.

* Spherical Bessel functions

$$\begin{cases} j_n(x) = \sqrt{\frac{\pi}{2x}}\, J_{n+\frac{1}{2}}(x) \\ y_n(x) = \sqrt{\frac{\pi}{2x}}\, Y_{n+\frac{1}{2}}(x) \end{cases} \qquad n \text{ integer.}$$

$$j_0(x) = \frac{\sin x}{x} \quad ; \quad y_0(x) = -\frac{\cos x}{x}$$

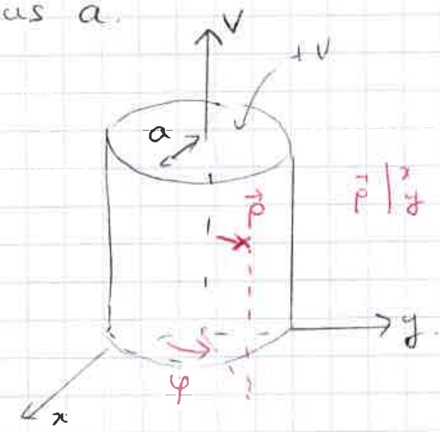
they are solutions of the D.E. :

$$x^2 y'' + 2xy' + (x^2 - (n+1)n)\, y = 0$$

5) Application : $\Delta\psi + k^2\psi = 0$ in the plane

This is a Schrödinger equation with constant potential, inside a disk of radius a. (also Helmholtz eq.)

$k^2 = cte$



$\vec{\rho}\begin{vmatrix} x \\ y \end{vmatrix} \Rightarrow \begin{vmatrix} \rho \\ \varphi \end{vmatrix}$ cylindrical coordinates

$$\Delta\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\varphi^2} = -k^2\psi$$

Search for separable solutions : $\psi = F(\rho).G(\varphi)$.

The DE becomes :

$$\frac{\rho}{F}\frac{d}{d\rho}(\rho F') + \frac{G''}{G} + k^2\rho^2 F = 0$$

$$\Rightarrow \frac{G''}{G} = -n^2 \qquad n \text{ real}$$

$$\Rightarrow G = e^{\pm in\varphi} \qquad n \text{ integer since } G \text{ periodic}$$

and for the radial part:

$$\frac{\rho}{F}(\rho F'' + F') + k^2\rho^2 = n^2$$

$$\Rightarrow \rho^2 F'' + \rho F' + (k^2\rho^2 - n^2)F = 0$$

Bessel: $x^2 y'' + xy' + (x^2 - \nu^2)y = 0$

change for $u = k\rho \Rightarrow \frac{dF}{du} = \frac{1}{k}\frac{dF}{d\rho}$

$$\Rightarrow \frac{d^2F}{d\rho^2} = k^2\frac{d^2F}{du^2}$$

$$\Rightarrow u^2\frac{d^2F}{du^2} + u\frac{dF}{du} + (u^2 - n^2)F = 0$$

$$\Rightarrow F(u) = A_n J_n(u) + B_n Y_n(u)$$

$$\Rightarrow F(\rho) = A_n J_n(k\rho) + B_n Y_n(k\rho)$$

So the general solution expresses as the linear combination

$$\boxed{\Psi(\rho,\varphi) = \sum_{n=-\infty}^{+\infty} \left[A_n J_n(k\rho) + B_n Y_n(k\rho)\right] e^{in\varphi}}$$

Bessel functions well adapted to cylindrical coordinates, just as Y_ℓ^m for spherical coordinates (angular part)

→ Ψ finite condition : if $\rho < a$ $B_n = 0$ since Y_n diverges at $\rho = 0$

→ symetry : if V is constant for $\rho < a$ $\Rightarrow$ Ψ indep of φ $\Rightarrow n = 0$

$$\Rightarrow \quad \Psi(\rho,\varphi) = A_0 J_0(k\rho)$$

→ boundary condition : Ψ vanishes at $\rho = a$

$$\Rightarrow \quad J_0(ka) = 0$$

gives conditions for k (quantification)

So $ka = \alpha_n$ with α_n the n-th root of Bessel J_0

$\Rightarrow k \simeq \frac{2,4}{a}, \frac{5,5}{a}, \frac{8,7}{a}$, etc... → ensemble de modes

+ Finding A_n ? necessary to introduce physical properties.

i.e. $$\iint_{disk} |\Psi|^2 \rho \, d\rho \, d\varphi = 1$$

then : $$2\pi \int_0^a A_0^2 \rho J_0^2(k_n \rho) d\rho = 1$$

the primitive for $x J_0^2(x)$ is found in the tables ;

we have : $$\int x J_0^2(x) dx = \frac{x^2}{2}\left[J_0(x)^2 - J_1(x)^2\right]$$

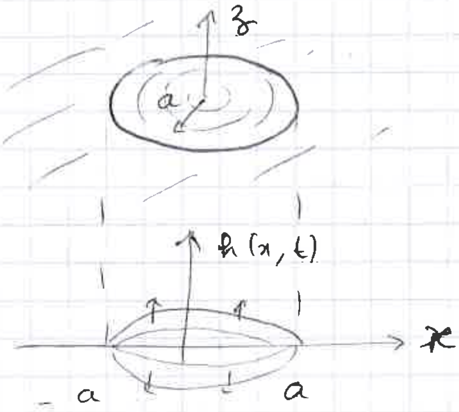
$$1 = \int_{\varphi=0}^{2\pi} \int_{\rho=0}^{a} A_0^2 \, \rho \, J_0^2(k_n \rho) d\rho$$

$$= \frac{2\pi A_0^2}{k_n^2} \int_0^{k_n a} u \, J_0^2(u) \, du$$

$$= \frac{2\pi A_0^2}{2 k_n^2}\left[u^2\left(J_0(u)^2 - J_1(u)^2\right)\right]_0^{k_n a} \qquad J_0(k_n a) = 0$$

$$= -\frac{\pi A_0^2}{k_n^2}(k_n a)^2 J_1(k_n a)^2$$ $\Rightarrow$ we have A_0

6) Application 2 : Stationnary waves inside a disc



combination of progressives cylindrical waves : stationnary waves

condition: amplitude null at $r=a$

$h(\vec{r},t)$ = vertical displacement of the surface

$\Rightarrow$ fundamental modes of a drum.

$h(x,t)$ follow the D.E :

$$\Delta h - \frac{1}{c^2}\frac{\partial^2 h}{\partial t^2} = 0 \qquad \text{(propagation equation)}$$

$\Rightarrow$ search separable solutions of the form:

$$h(\vec{r},t) = \psi(r,\varphi)\cdot g(t)$$

(r,φ): cylindrical coordinates.

$$\Delta h = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial h}{\partial \rho}\right) \quad ; \quad \frac{\partial^2 h}{\partial t^2} = \psi \cdot g''(t)$$

$$\Rightarrow \quad g\,\Delta\psi - \frac{\psi}{c^2}g'' = 0 \quad \Rightarrow \quad \frac{\Delta\psi}{\psi} = \frac{g''}{g c^2}$$

- equation in g: $\dfrac{g''}{g c^2} = -k^2 \Rightarrow g'' + k^2c^2 g = 0$

$$g(t) = e^{\pm i\omega t} = \begin{cases}\cos\omega t \\ \sin\omega t\end{cases} \quad \text{with } \omega = kc$$

(real solutions)

- equation in ψ:

$$\frac{\Delta\psi}{\psi} = -k^2 \Rightarrow \Delta\psi + k^2\psi = 0$$

already solved in the previous example.

$$\psi(r,\varphi) = \sum_{n=-\infty}^{+\infty} A_n J_n(k\rho)\, e^{in\varphi}$$

and

$$h(r,\varphi,t) = \sum_{n=-\infty}^{+\infty} A_n J_n(k\rho)\, e^{in\varphi}\, e^{i\omega t}$$

that can be expressed, since h must be real as the combination:

$$h(r,\varphi,t) = \sum_{\substack{n=0 \\ p=1}}^{\infty} \underbrace{h_{np}\, J_n\left(\alpha_{np}\frac{\rho}{a}\right) \cos(n\varphi + \varphi_{onp}) \cos(\omega_{np} t + \tau_{np})}_{\text{"mode"}}$$

h_n: coefficient ("amplitude" of the mode)

$\alpha_{n,p}$: p-th zero of J_n (needed by the boundary cond. $h=0$ at $r=a$)

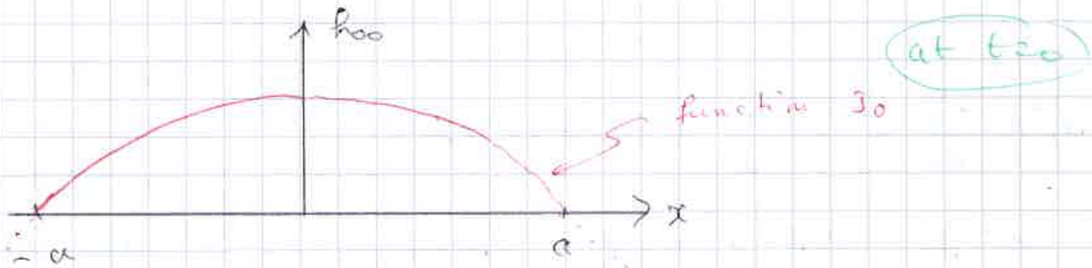
φ_{on} : origin of φ for each mode (depends upon the initial condition)

τ_n : origin of t $\omega_{np} = c\, k_{n,p}$ pulsation of the mode

mode 00 : $h_{oo} = J_0\left(k_{oo}\frac{\rho}{a}\right)\cos(\omega_0 t + \tau_0)$

$\alpha_{oo} = 2{,}4048$ (1^{st} 0 of J_0)

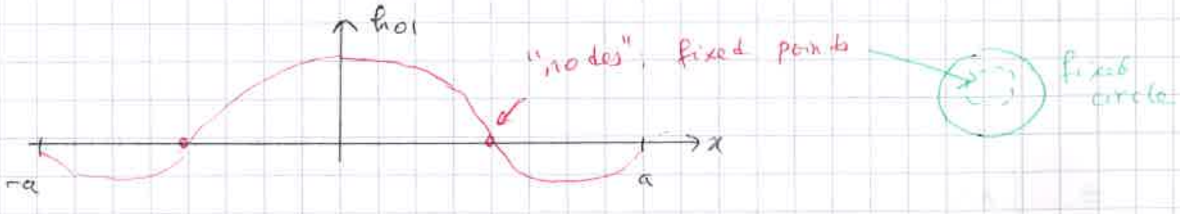
$\omega_0 = ck = 2{,}4048\, c.$



mode 01 : $h_{01} = J_0\left(k_{01}\frac{\rho}{a}\right)\cos(\omega_1 t + \tau_1)$

$\alpha_{01} = 5{,}52$ (2^{nd} 0 of J_0)

$\omega = ck = 5{,}52\, c$ (2 times faster)



Show the anim on wikipedia, vibration of a circular membrane (look for drum vibration bessel on google)

§ 2-h : orthogonality of J_ν

We define the scalar product

$$\langle f, g \rangle = \int_0^1 x\, f(x)\, g(x)\, dx$$

for functions square summable over $[0, 1]$

We have :

$$\boxed{\begin{aligned} \langle J_\nu(\alpha_n x), J_\nu(\alpha_p x)\rangle &= \int_0^1 x\, J_\nu(\alpha_n x) \cdot J_\nu(\alpha_p x)\, dx \\ &= 0 \quad \text{if } n \neq p \\ &= \tfrac{1}{2} J_{\nu+1}^2(\alpha_n) = \tfrac{1}{2} J_\nu'^2(\alpha_n) \quad \text{if } n = p \end{aligned}}$$

$\alpha_n = n^{th}$ zero of J_ν

Proof: (as an exercice)

$J_\nu(x)$ verifies the Bessel D.E

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0$$

$x = \alpha_n X$; $y = J_\nu(x) = J_\nu(\alpha_n X)$

$$\frac{dy}{dx} = \frac{dy}{dX} \cdot \frac{dX}{dx} = \frac{1}{\alpha_n} \frac{dy}{dX}$$

$$\alpha_n^2 X^2 \frac{d^2y}{dx^2} + \alpha_n X \frac{dy}{dx} + (\alpha_n^2 X^2 - \nu^2) y = 0.$$

(2) $$X^2 y'' + X y' + (\alpha_n^2 X^2 - \nu^2) y = 0$$

$\left[\, ' \equiv \frac{d}{dX} \right]$

let $\begin{cases} u(X) = J_\nu(\alpha_n X) \\ v(X) = J_\nu(\alpha_p X) \end{cases}$

$$\Rightarrow \begin{cases} X^2 u'' + X u' + (\alpha_n^2 X^2 - \nu^2) u = 0 & \times v \\ \underbrace{X^2 v'' + X v'}_{X \frac{d}{dX}(X v')} + (\alpha_p^2 X^2 - \nu^2) v = 0 & \times u \end{cases}$$

$$\Rightarrow \begin{cases} X v \frac{d}{dX}(X u') + u v (\alpha_n^2 X^2 - \nu^2) = 0 \\ X u \frac{d}{dX}(X v') + u v (\alpha_p^2 X^2 - \nu^2) = 0 \end{cases}$$

divide $/ X$

$$\underbrace{v \frac{d}{dX}(X u')}_{1^{st} \text{ term of the derivative of } X u' v} - u \frac{d}{dX}(X v') + u v X (\alpha_n^2 - \alpha_p^2) = 0$$

$$v \frac{d}{dX}(X u') = \frac{d}{dX}(X u' v) - X u' v'$$

hence :

$$\frac{d}{dx}(x u' v) - \frac{d}{dx}(x u v') + u v x (\alpha_n^2 - \alpha_p^2) = 0.$$

$$\left[x u' v - x u v' \right]_0^1 + \int_0^1 (\alpha_n^2 - \alpha_p^2)\, u \cdot v \; x \, dx = 0 \qquad (1)$$

↳ vanishes at 0 (because of x)

vanishes at 1 because $v(1) = J_\nu(\alpha_p) = 0$ by def of $\alpha_{n,p}$
$u(1) = J_\nu(\alpha_n) = 0$

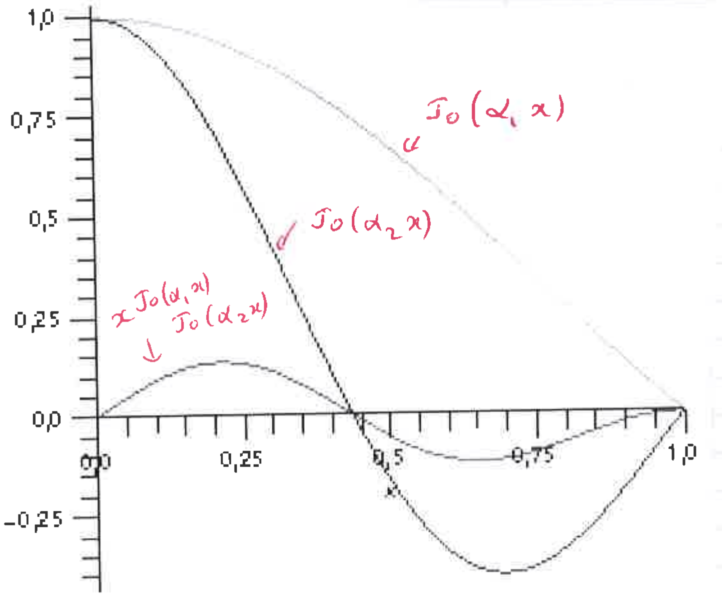
Therefore : $\int_0^1 (\alpha_n^2 - \alpha_p^2)\, u . v . \, dx = 0$

$$\Rightarrow \quad \boxed{\int_0^1 x \, J_\nu(\alpha_n x) . J_\nu(\alpha_p x) = 0}$$

orthogonality relation, between $J_\nu(\alpha_n x)$ and $J_\nu(\alpha_p x)$

same function

≠ scale



(file: orthogonality_J0 . gif)

→ finding the norm :

in (1) differentiate with respect to α_n

and set $\alpha_p = \alpha_n$ in the result.

equivalent to use l' Hôpital rule:

$$\int_0^1 u . v . \, dx = \frac{\left[x u' v - x u v' \right]_0^1}{(\alpha_n^2 - \alpha_p^2)} = \frac{\frac{d}{d\alpha_n} x \left[\alpha_n J_\nu'(\alpha_n) J_\nu(\alpha_p) - \alpha_p J_\nu(\alpha_n) J_\nu'(\alpha_p) \right]}{\frac{d}{d\alpha_n} (\alpha_n^2 - \alpha_p^2)}$$

5F30

one obtains: $\boxed{\int_0^1 x\, J_\nu^2(x)\, dx = \frac{1}{2} J_\nu'(\alpha_n)^2 = \frac{1}{2} J_{\nu+1}(\alpha_n)^2}$

use of: $\nu J_\nu' = \nu J_\nu - x J_{\nu+1}$

* **Fourier - Bessel Series**

since $J_\nu(\alpha_n x)$ are orthogonal

we can write, for functions f with $f(1) = 0$

$$\boxed{f(x) = \sum_{n=1}^{\infty} c_n J_\nu(\alpha_n x)}$$

n número de zero

with

$$\boxed{c_n = \frac{\int_0^1 x\, f(x)\, J_\nu(\alpha_n x)}{\frac{1}{2} J_\nu'(\alpha_n)^2}}$$